

# The Black-Scholes Equation as an Optimality Condition

Michael J. Best<sup>1</sup> and Thamayanthi Chellathurai<sup>2</sup>

<sup>1</sup>Department of Combinatorics and Optimization

University of Waterloo

Waterloo

<sup>2</sup> Treasury and Risk Management

Canadian Imperial Bank of Commerce

Toronto

In an ideal complete market, options are redundant since any contingent claim, written on an underlying risky asset, can be replicated perfectly. However, in practice it is not possible to hedge the risk perfectly, or, the underwriters of the contingent claims may take risks by not hedging the risk perfectly. In this paper, we study the optimal hedging problem using stochastic optimal control theory. An analytical solution is obtained by solving the resultant Hamilton-Jacobi-Bellman equation. It is shown that the optimal strategy coincides with the Black-Scholes' replication strategy when the underwriter is willing to allocate the required capital. The optimal strategy depends on the instantaneous rate of return of the underlying risky asset when the underwriter does not have the required capital.