1. **Prove König’s Theorem**: the size of a largest matching in a bipartite graph $G$ is equal to the size of a smallest vertex cover of the edges.

2. **Prove Brooks’ Theorem**: if $G$ is a connected graph with maximum degree $\Delta$, then either $G$ has a proper colouring of its vertices with $\Delta$ colours, or $G$ is either a complete graph or a cycle.

3. **Prove Turán’s Theorem**: if $G$ is a simple graph $G$ with $n$ vertices and $r$ is a positive integer such that no subgraph of $G$ is isomorphic to $K_{r+1}$, then $|E(G)| \leq |E(T_{n,r})|$ and equality holds if and only if $G = T_{n,r}$. (Here $T_{n,r}$ is the complete multipartite graph with $n$ vertices, $r$ parts, and each part has size either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$.)

4. Let $G$ be a (multi)graph of maximum degree $\Delta$. Let $e = v_0v_1$ be an edge of $G$. Let $k \geq \Delta + 1$ and suppose that $G$ is not $k$-edge-colourable but that $G - e$ is $k$-edge-colourable. Let $\phi$ be a $k$-edge-colouring of $G - e$. For all $u \in V(G)$, let $\phi(u)$ denote the subset of $[k]$ that does not appear (in $\phi$) on any edge incident with $u$.

Suppose that $P = v_0v_1 \ldots v_m$ is a path such that for all $i \geq 1$, there exists $j < i$ such that the colour $\phi(v_i v_{i+1})$ does not appear at $v_j$ (i.e. is in $\phi(v_j)$).

**Prove**: For every $i \neq j$, the set of colours not appearing at $v_i$ is disjoint from the set of colours not appearing at $v_j$. That is, prove that, for every $i \neq j$, $\phi(v_i) \cap \phi(v_j) = \emptyset$.

*Hint: Use double induction, first on $m$, then on $m - j$ where $v_j$ is missing a colour that is also missing at $v_m$ (i.e. $\phi(v_m) \cap \phi(v_j) \neq \emptyset$). Use Kempe changes! It may be useful in some cases to also consider a second colour, in particular one missing at $v_{j+1}$."

5. Let $H$ be a subgraph of a graph $G$. A walk $W$ in $G$ is $H$-avoiding if no edge of $W$ and no internal vertex of $W$ is in $H$. Define the relation $\sim$ on $E(G) \setminus E(H)$ by $e \sim e'$ if there is an $H$-avoiding walk in $G$ containing both $e$ and $e'$.

**Prove** that $\sim$ is a transitive relation.

6. Let $F$ be a collection of subtrees of a tree $T$, and let $k$ be a positive integer.

**Prove** at least one of the following holds:

(a) there are $k$ vertex disjoint trees in $F$, or
(b) there is a set $X$ of $< k$ vertices in $T$ that intersects each tree in $F$.

*Hint: Consider the set of all edges $e$ in $T$ such that both components of $T - e$ contain a tree in $F$; prove that this set forms a subtree of $T$; then consider a leaf of this subtree."