COMPREHENSIVE EXAM: ENUMERATION

1:00–4:00 pm, June 18, 2004

1(a) For fixed integer $m \geq 2$, let a_n denote the number of partitions of the integer n such that no part is divisible by m. Let b_n denote the number of partitions of the integer n such that every part occurs at most $m-1$ times. Show that for every nonnegative integer n, $a_n = b_n$.

(b) Let c_n denote the number of partitions of the integer n such that every even part occurs at most three times. Let d_n denote the number of partitions of the integer n such that every part occurs at most seven times. Show that for every nonnegative integer n, $c_n = d_n$.

2(a) Show that

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$

(b) Show that for every nonnegative integer n,

$$\sum_{j=0}^{n} \binom{2j}{j} \binom{2n-2j}{n-j} = 4^n.$$

(c) Show that for every nonnegative integer n,

$$\sum_{j=0}^{n} \binom{2j}{j} \binom{j}{n-j} (-1)^{n-j} = 2^n.$$

3. For each $n \geq 0$, let c_n be the number of compositions of n in which there are no consecutive pairs of even parts.

(a) Show that

$$\sum_{n=0}^{\infty} c_n x^n = \frac{1-x^2}{1-x-2x^2+x^4}.$$

(b) Use (a) to give a linear recurrence together with sufficient initial conditions to uniquely determine $\{c_n,\ n \geq 0\}$.

(c) Give a direct proof of the recurrence in (b) by describing an appropriate bijection for these compositions.

4. For a positive integer k, a *k-ary rooted tree* (k-RT) is defined recursively as follows:

- every k-RT has a root node;
- every node of a k-RT may or may not have one child node of each of k types;
- every k-RT has only finitely many nodes.

(For example, the case $k = 2$ gives the definition of binary rooted trees.)

(a) Show that the number of k-ary rooted trees with n nodes is

$$\frac{1}{n}\binom{kn}{n-1}.$$

(b) A *terminal node* of a k-RT is a node with no children. Among all k-RTs with n nodes, what is the average number of terminal nodes per tree?

5(a) Show that for every natural number n, there are n^{n-2} labelled trees on the vertex set $\{1, 2, ..., n\}$.

(b) Show that for every natural number n,

$$\sum_{j=0}^{n-1}\binom{n}{j} j^j (n-j)^{n-j-1} = n^n.$$

(c) Determine the number of labelled trees on the vertex set $\{1, 2, ..., n\}$ such that vertex 1 is adjacent to vertex 2.

(d) Determine the number of rooted labelled trees on the vertex set $\{1, 2, ..., n\}$ in which the root vertex has degree k, for fixed positive integer k.