Provide solutions to at least five of the seven questions.

Greater credit will be given to well-reasoned and complete solutions than to fragments or sketches of solutions.

1. A trivalent labelled tree is a tree with vertex-set \{1, \ldots, n\} for some positive integer \( n \) in which each vertex has degree either 1 or 3. Show that if \( n = 2k \) then the number of trivalent labelled trees with \( n \) vertices is

\[
\frac{(2k - 2)!}{2^{k-1}} \binom{2k}{k-1}.
\]

2. (a) Fix an integer \( b \geq 2 \). Let \( \mathcal{A} \) be the set of integer partitions in which each part occurs at most \( b - 1 \) times. Let \( \mathcal{B} \) be the set of integer partitions in which no part is divisible by \( b \). Show that for every integer \( n \), the number of partitions of \( n \) in \( \mathcal{A} \) equals the number of partitions of \( n \) in \( \mathcal{B} \).

(b) In the special case \( b = 2 \), describe explicitly a weight-preserving bijection between the sets \( \mathcal{A} \) and \( \mathcal{B} \) in part (a). (A proof of correctness is not required.)

(c) Provide a bijection as in part (b) for the general case \( b \geq 2 \).

3. Let \( \gamma_k = [t^k] \prod_{i \geq 1} (1 - tx_i)^{-1} \) where \( t, x_1, x_2, \ldots \) are indeterminates.

(a) Let \( \phi(\gamma_1, \gamma_2, \ldots) \) be a formal power series in \( \gamma_1, \gamma_2, \ldots \). Prove that

\[
[x_1 \cdots x_n] \phi(\gamma_1, \gamma_2, \ldots) = \left[ \frac{x^n}{n!} \right] \phi \left( \frac{x}{1!}, \frac{x^2}{2!}, \ldots \right).
\]

(b) Prove that the ordinary generating series for the number of alternating sequences of even length is

\[
\left( \sum_{k \geq 0} (-1)^k \gamma_k \right)^{-1},
\]

and thence find the generating series for the number of alternating permutations of even length.
4. A proper 2-cover of order $k$ of $\{1, \ldots, n\}$ is a set $B$ of non-empty and pairwise mutually distinct subsets $B_1, \ldots, B_k$ of $\{1, \ldots, n\}$ such that each element of $\{1, \ldots, n\}$ appears in exactly two members of $B$. Prove that the number $a(k, n)$ of such covers is given by

$$A(x, y) = \sum_{k,n \geq 0} \frac{x^k y^n}{k! n!} \binom{n}{k} \binom{k}{2} a(k, n)$$

where

$$A(x, y) = e^{-x^{2}(e^y - 1)/2 - x} \sum_{k,n \geq 0} \frac{x^k y^n}{k! n!} \binom{n}{k} \binom{k}{2}$$

5. (a) Let $\alpha$ and $x$ be indeterminates. Find a formal power series $f(y)$ such that $e^{\alpha y} = f(\alpha e^{-x})$.

(b) Let $\beta$ be another indeterminate. From part (a) or otherwise, prove that

$$(\alpha + \beta)(n + \alpha + \beta)^{n-1} = \alpha \beta \sum_{k=0}^{n} \binom{n}{k} (k + \alpha)^{k-1} (n - k + \beta)^{n-k-1}.$$

6. (a) An inversion of a permutation $a_1 a_2 \ldots a_n$ is a pair of indices $(i, j)$ such that $1 \leq i < j \leq n$ and $a_i > a_j$. Let $\text{inv}(\sigma)$ denote the number of inversions of the permutation $\sigma \in S_n$, and for an indeterminate $q$ define the polynomial

$$[n]_q! := \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)}.$$

Give a combinatorial proof that for $n \geq 1$,

$$[n]_q! = [n - 1]_q! (1 + q + q^2 + \cdots + q^{n-1}).$$

(b) Let $q = p^c$ be a prime power. Let $V$ be an $n$-dimensional vector space over the finite field $\text{GF}(q)$. Show that the number of ordered bases of $V$ is

$$(q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1}).$$

(c) Show that the number of $k$-dimensional subspaces of $V$ is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!},$$

with the notation defined in part (a).

7. (a) Let $Z_{S_n}(x_1, x_2, \ldots, x_n)$ denote the cycle index polynomial of the symmetric group $S_n$, for each $n \geq 0$. Give a formula for the generating series

$$F(t; x_1, x_2, \ldots) = \sum_{n=0}^{\infty} t^n Z_{S_n}(x_1, x_2, \ldots, x_n).$$

(b) Let $c_n$ denote the number of rooted (but unlabelled) trees on $n$ vertices. Using part (a) or otherwise, find an expression giving each $c_n$ in terms of $c_0, c_1, c_2, \ldots, c_{n-1}$. 

2