GRAPH THEORY COMPREHENSIVE

1 June 2014, 1:30-4:30 p.m.

Examiners: Jim Geelen and Luke Postle

Instructions: A complete examination consists of solutions to all six questions. All questions have equal value and, in multipart questions, all parts have equal value. All proofs should be derived from first principles unless stated otherwise.

Problem 1. Prove Petersen's theorem that each 3-connected cubic graph has a perfect matching. (You may use Tutte's matching theorem, but if you do, you should state that theorem explicitly.)

Problem 2. Prove that in every 2-connected graph there is a basis for the cycle space that has at most one odd cycle. (Hint: consider an ear decomposition starting from an odd cycle.)

Problem 3. State and prove Vizing's edge-colouring theorem for simple graphs.

Problem 4. Prove that every simple d-regular vertex-transitive graph is d-edge-connected.

Problem 5. The graph H is the tree with degree-sequence $(3, 2, 1, 1, 1)$.

(a) Prove that, if G is a simple connected graph that does not contain a subgraph isomorphic to H, then either $|V(G)| \leq 4$ or G has maximim degree at most 2 or G is a star.

(b) For each integer $n \geq 0$, determine the maximum number, $\mathrm{ex}(n, H)$, of edges in a simple n-vertex graph that has no subgraph isomorphic to H.

Problem 6. Prove that every 3-edge-connected graph has a $(\mathbb{Z}_2 \times \mathbb{Z}_3)$-flow. (You may use a version of Menger's Theorem, but, if you do, you should state that theorem explicitly.)