1 Linear Programming and Complementarity

Let \( A \in \mathbb{R}^{m \times n} \) be a given matrix, and \( b \in \mathbb{R}^m \), \( c \in \mathbb{R}^n \) given vectors. Let \( (P), (D) \) stand for the following pair of primal-dual linear programming problems:

\[
\begin{align*}
(P) & : \quad \min_{x} c^T x \quad \text{s.t.} \quad Ax = b, \quad x \geq 0, \\
(D) & : \quad \max_{y} b^T y \quad \text{s.t.} \quad A^T y + s = c, \quad s \geq 0.
\end{align*}
\]

Recall that we say that a pair of feasible points \((x, (y, s))\) is \textit{complementary} if \( x^T s = 0 \).

1. State a theorem that relates complementarity to optimality.

2. A pair of feasible points is said to be \textit{strictly complementary} if for each \( i = 1, \ldots, n \), \( x_i s_i = 0 \) and \( x_i + s_i > 0 \). Argue that if \((x^*, (y^*, s^*))\) is a feasible strictly complementary solution, then for any optimizer \( \hat{x} \) of the primal, \( \text{supp}(\hat{x}) \subset \text{supp}(x^*) \). Here, “\( \text{supp}(x) \)” denotes the indices (i.e., a subset of \( \{1, \ldots, n\} \)) of nonzero entries of \( x \). [Hint: \( \hat{x} \) must also be complementary with \((y^*, s^*)\).]

3. Say that \( B \cup N \) (both \( B, N \) are subsets of \( \{1, \ldots, n\} \)) is a \textit{strict complementarity partition} of \( \{1, \ldots, n\} \) for the above LP if there is a strictly complementary solution.
such that \( \text{supp}(x^*) = B \) while \( \text{supp}(s^*) = N \). Show that the strict complementarity partition is uniquely determined by the LP, i.e., it cannot have two distinct strict complementarity partitions.

2 Convex Functions

Say that a convex function \( f : \mathbb{R}^n \to \mathbb{R} \) is strongly convex with modulus \( \mu > 0 \) if the function \( g(x) \equiv f(x) - \frac{\mu}{2} \| x \|^2 \) is convex.

1. Show that an equivalent definition is: for all \( x, y \in \mathbb{R}^n, \lambda \in [0, 1], \)

\[
f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y) - \frac{1}{2}\mu \lambda (1 - \lambda)\|x - y\|^2.
\]

[Hint: write \( \| (1 - \lambda)x + \lambda y \| \) in terms of \( \lambda \) and \( \| x \|^2, \| y \|^2 \) and \( \| x - y \|^2 \).]

2. Suppose also that \( f \) is differentiable. Show that strong convexity implies that

\[
(\nabla f(x) - \nabla f(y))^T (x - y) \geq \mu \| x - y \|^2,
\]

for all \( x, y \).

3. Consider the function \( f(x) = \| Ax - b \|^2 \), where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). Suppose that \( \text{rank}(A) = n \). Show that \( f \) is strongly convex.

3 Second Order Cone

The second-order cone \( C_2^n \) is defined to be

\[
C_2^n = \left\{ x \in \mathbb{R}^n : x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2} \right\}.
\]

Second-order cone programming (SOCP) in standard form means minimizing \( c^T x \) subject to \( Ax = b, x \in C_2^{n_1} \times \cdots \times C_2^{n_r} \), where \( c \) is a given \( n \)-vector, \( A \) is a given \( m \times n \) matrix, \( b \) is a given \( m \)-vector, and \( n_1 + \cdots + n_r = n \) so that the containment makes sense.

1. Show that the second-order cone is a convex set.

2. Define the following function \( \Phi \) that maps \( \mathbb{R}^n \) (vectors) into \( S^n \) (\( n \times n \) symmetric matrices):

\[
\Phi(x) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_1 \\ \vdots & \ddots \\ x_n & \cdots & x_1 \end{bmatrix}.
\]
Entries of $\Phi(x)$ outside the main diagonal and first row and column are zeros.

Show that $x \in C_2^n \iff \Phi(x) \succeq 0$. [Hint: One possible approach is to consider products of the form $v^T \Phi(x)v$; use a carefully chosen $v$ for one direction and the Cauchy-Schwarz inequality for the other. Another possible approach is to use a characterization of positive definiteness in terms of Cholesky factorization applied to a reordering of $\Phi(x)$.

3. The result of Item 2 is usually cited to justify the statement that: second-order cone programming is a special case of semidefinite programming. Explain.

4 Optimality Conditions

For this question, assume that $f, g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$, are convex differentiable functions and that $\bar{x} \in \Omega \subseteq \mathbb{R}^n$; $\Omega$ a convex set.

1. (a) Define the tangent cone of $\Omega$ at $\bar{x}$, and denote it by $T_\Omega(\bar{x})$. [Hint: You can take advantage of the convexity of $\Omega$.]

(b) Define the nonnegative polar of a convex cone $K$ and denote it by $K^*$.  

2. State and prove a characterization of optimality for

$$\bar{x} \in \text{argmin}_{x \in \Omega} f(x).$$

(The necessity part for general functions $f$ and general sets $\Omega$ is sometimes called the Rockafellar-Pshenichnyi condition.)

3. Consider the following constrained convex optimization problem with $\Omega = \mathbb{R}^n$.

\[
\text{(CP)} \quad \begin{aligned}
\min_x & \quad f(x) \\
\text{s.t.} & \quad g_i(x) \leq 0, \quad i = 1, \ldots, m.
\end{aligned}
\]

(a) State the Lagrangian function $L(x, \lambda)$ for the constrained problem (CP).

(b) Define the linearizing cone for the constraints in (CP) at a feasible point $\bar{x}$. Find and prove the relationship between this linearizing cone and the tangent cone of the feasible set, i.e., a subset condition.

(c) State the Karush-Kuhn-Tucker optimality conditions and prove it under a weakest constraint qualification obtained from the cone conditions in Item 3b, i.e., that the subset condition holds with equality.

5 Duality and Supporting Hyperplanes

Let the $f, g_i$ and set $\Omega$ be as above in Section 4. Consider the above constrained convex program (CP) with the additional set constraint

\[
\text{(CPS)} \quad \begin{aligned}
\min_x & \quad f(x) \\
\text{s.t.} & \quad g_i(x) \leq 0, \quad i = 1, \ldots, m \\
x \in \Omega.
\end{aligned}
\]
1. State the Slater condition first for (CP) and then for (CPS).

2. State the Lagrangian dual of (CPS). Prove that weak duality holds.

3. Consider the convex program

\[
\text{(CPSS)} \quad \min_x \quad f(x) := e^{-\sqrt{x_1x_2}} \\
\text{s.t.} \quad x_1 = 0, \quad x \in \Omega := \mathbb{R}^2_+.
\]

Here \( f \) is defined as \(+\infty\) outside of \( \Omega \).

   (a) Derive the Lagrangian dual of (CPSS).
   (b) Find the optimal primal and dual values. Explain your results.

6  Constrained Optimization

Let

\[
f : \mathbb{R}^n \to \mathbb{R}, \ g : \mathbb{R}^n \to \mathbb{R}^m, \ h : \mathbb{R}^n \to \mathbb{R}^p
\]

be differentiable functions. Consider the general constrained nonlinear program

\[
\text{(NLP)} \quad \min_x \quad f(x) \\
\text{s.t.} \quad g(x) \leq 0 \quad \text{and} \quad h(x) = 0.
\]

1. State the Mangasarian-Fromovitz constraint qualification for (NLP) at a feasible point \( \bar{x} \).

2. State the appropriate KKT conditions under a constraint qualification at a feasible point \( \bar{x} \).

3. Consider the above (NLP) with only equality constraints \( h(x) = 0 \). (There are no inequality constraints.)

   (a) Construct an augmented Lagrangian for the nonlinear program (NLP).
   (b) Relate an unconstrained stationary point of the augmented Lagrangian to the original nonlinear program.
   (c) Give one step of the augmented Lagrangian algorithm, or any other algorithm that utilizes the augmented Lagrangian formulation.