Department of Combinatorics and Optimization

CONTINUOUS OPTIMIZATION COMPREHENSIVE

July 2003: **3 hours**

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Instructions: Answer as many questions as you can. (There are 6 questions.) Complete answers are preferred over fragmented ones.

1 LP

[10 marks]

Let A be an (m,n) matrix and b be an $m-$vector. Suppose the set $R = \{x \mid Ax \leq b\}$ is nonempty. Prove that R has extreme points if and only if $\text{rank}(A) = n$.

2 Theorem of Gordan

[15 marks]

Prove the following theorem. State carefully the results you use, e.g. the hyperplane separation theorem or Farkas' Lemma.

Theorem 2.1 *(Theorem of Gordan) Let A be a $m \times n$ matrix, $x \in \Re^n, u^T \in \Re^m$. Then one and only one of the following conditions holds:*

1. *there exists x such that $Ax < 0$*
2. *there exists $u \neq 0$ such that $uA = 0$ and $u \geq 0$.*

3 Approximation of KKT Multipliers

[20 marks]

Consider the (sufficiently smooth) nonlinear inequality constrained program

$$(NLP) \qquad \begin{array}{rl} \min & f(x) \\ \text{subject to} & g(x) \leq 0 \in \Re^m \\ & x \in \Re^n \end{array}$$

1. Describe the steps of the *exterior penalty method* for this NLP.
2. At each (major) iteration of the method, derive an approximation for the optimal KKT multipliers.
3. Under what conditions does this method provide an optimum of NLP?
4. Use the penalty method to prove the Karush-Kuhn-Tucker Theorem for this problem. (Carefully state the theorem you are proving and the constraint qualification/regularity condition you are using.)

4 Fractional Programming

[10 marks]

Consider the program

$$(FP) \qquad \begin{array}{rl} \min & f(x)/g(x) \\ \text{subject to} & x \in X \subset \Re^n, \end{array}$$

where $g(x) > 0,\ \forall x \in X$. For $\lambda \in \Re$, Define

$$Q(\lambda) = \min_{x \in X} \{f(x) - \lambda g(x)\},$$

and suppose that a scalar λ^* and a vector x^* satisfy $Q(\lambda^*) = 0$ and

$$x^* \in \arg\min_{x \in X} \{f(x) - \lambda^* g(x)\}.$$

Show that x^* is an optimal solution of the original problem.

5 Minimax Problem

[10 marks]

Consider the problem

$$\min_{x \in \Re^n} \ \max \ \{g_1(x), \ldots, g_r(x)\},$$

where g_j are continuously differentiable.

1. Show that if x^* is a local minimum, then there exists a vector $\mu^* = (\mu_1^*, \ldots, \mu_r^*)$ such that

$$\sum_{j=1}^{r} \mu_j^* \nabla g_j(x^*) = 0, \quad \mu^* \geq 0, \quad \sum_{j=1}^{r} \mu_j^* = 1,$$
$$\mu_j^* = 0, \text{ if } g_j(x^*) < \max\{g_1(x^*), \ldots, g_r(x^*)\}.$$

2. Was a constraint qualification needed to prove Item 1? If yes, provide an example where Item 1 fails.

6 Optimality Conditions

[15 marks]

Consider the parametric quadratic programming problem

$$\min\{-t\mu^T x + \frac{1}{2}x^T C x \mid x_1 + \cdots + x_n = 1, \ \ x \geq 0\}, \qquad (6.1)$$

where C is an (n, n) positive semi-definite matrix, μ is an $n-$vector, and t is a scalar parameter. Let $C = [\sigma_{ij}]$ and assume that k satisfies $\mu_k > \mu_i, \ \forall i \neq k$. Define x^* to be the n–vector having all components zero except for the k–th which has value 1. Use the optimality conditions for (6.1) to show that x^* is optimal for (6.1) for all t satisfying

$$t \geq \max\{\frac{\sigma_{kk} - \sigma_{ik}}{\mu_k - \mu_i} \mid i = 1, \ldots, n, \ \ i \neq k\}.$$