C&O — CONTINUOUS OPTIMIZATION

COMPREHENSIVE EXAM — Summer 2018

Thursday, June 18, 2018, 1:00 pm to 4 pm (3 hours), MC 2018B

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The exam has 4 problems on 3 pages.

Contents

1 Unconstrained Minimization

Consider the unconstrained minimization

$$p^* = \min_{x \in \mathbb{R}^n} f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a sufficiently smooth function. Let x_c denote a current approximation for a minimum of f, if it exists.

1.1 Coercivity

1. Define: f is a *coercive* function on $\mathbb{R}^n$.

2. Suppose that f is coercive. Prove that p^* exists and is attained.

3. Give examples of quadratic functions $q(x)$ on $\mathbb{R}^n$ that are: (i) coercive and (ii) not coercive. (For not coercive, ensure that the function is not linear.)

1.2 First Order Model and Steepest Descent

1. State the first order model for f at x_c and use it to derive the steepest descent direction at x_c.
2. Suppose that f is a strictly convex quadratic function. Explicitly find x_+, the next step in Cauchy's steepest descent method with *exact line search*.
3. Define order one convergence.
4. State an error result for the improvement in the objective function when going from x_c to x_+.

1.3 Second Order Model and Newton's Method

1. State the second order model for f at x_c and use it to derive Newton's Method from x_c.
2. State and prove a sufficient condition that guarantees that the Newton direction is a descent direction.
3. Define order two convergence.
4. Show by either a direct argument or by correctly citing a general-purpose theorem that Newton's method for minimizing the convex function

   $$f(x) = \frac{x^3}{3} - ax$$

   over the open interval $(0, 2\sqrt{a})$ converges to the minimizer with order 2 for any starting point in that interval.

2 Quadratic-Quadratic Program, QQP

Consider the program

$$(\textbf{QQP}) \qquad \min \operatorname{trace} AXBX^T \text{ s.t. } X^TX = I,$$

where $A, B \in \mathbb{S}^n$, the space of $n \times n$ symmetric matrices and $X \in \mathbb{R}^{n\times n}$.

1. State an appropriate constraint qualification for **QQP**. Show that it holds for any X in the feasible region.
   (Note that the constraints should be understood as $n(n+1)/2$ equalities corresponding to the upper triangular entries of the product X^TX.)
2. Write down the Lagrangian function.
3. State the optimality conditions (including stationarity conditions) that can be applied here.

3 Fenchel Conjugate

Let $h : E \to (-\infty, \infty]$, be an extended valued function on the Euclidean space E.

1. Define:
   (i) the epigraph of h;
   (ii) h is closed;
   (iii) h is lower semicontinuous;
   (iv) h is convex;
   (v) the Fenchel conjugate, h^*.

2. Determine the Fenchel conjugate of:
   (i) $x^T Ax + b^T x$ when A is a symmetric positive definite matrix;
   (ii) $f(x) = |x|$ (absolute value);
   (iii) $f(x) = ||x||_1$ (1-norm).

4 Linear Programming, LP

1. State the linear program in*standard form* with n variables and m constraints.

2. State the definition of a basic feasible solution, **BFS**. Using m, n, state how many possible **BFS** s the **LP** can have.

3. Find the dual program. State and prove the weak duality theorem.

   State the complementary slackness condition for optimality, and prove that if complementary slackness holds, then both the primal and dual optimizers are attained with the same values.

4. Can you find an **LP** where the primal and dual optimal values, $p^*, d^* \in [-\infty, +\infty]$, are *not* equal? Why?