COMBINATORICS AND OPTIMIZATION

RESEARCH REPORTS

January - December 1995

This is a list of abstracts of Research Reports published by the Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1.

Copies of individual papers may be obtained by writing directly to the author(s) at the above address.

Addresses for those persons marked with an asterisk are given at the end.

95-01 **Euclidean Geometry of Distance Regular Graphs**

C. D. Godsil

A graph is distance regular if it is connected and, given any two vertices u and v at distance i, the number of vertices x at distance j from u and k from v is determined by the triple (i, j, k). Distance regular graphs are interesting because of their connections with coding theory, design theory and finite geometry.

We can introduce geometric methods into the study of distance regular graphs as follows. Let X be a graph on n vertices and identify the i-th vertex of X with the i-th standard basis vector e_i in $\mathbb{R}^n$. Let θ be an eigenvalue of (the adjacency matrix of) X and let U be the corresponding eigenspace. We then associate to the i-th vertex of X the image of e_i under orthogonal projection onto U. If U has dimension m then we have a mapping from $V(X)$ into $\mathbb{R}^n$. If X is distance regular then it can be shown that the image of $V(X)$ lies in a sphere centred at the origin, and that the cosine of the angle between the vectors representing two vertices u and v is determined by the distance between them in X. In this paper we survey some of the applications of these methods.

95-02 **Singularities of Toric Varieties associated with Finite Distributive Lattices**

David G. Wagner

With each finite lattice L we associate a projectively embedded scheme $V(L)$; as Hibi has shown, the lattice D is distributive if and only if $V(D)$ is irreducible, in which case it is a toric variety. We first apply Birkhoff's structure theorem for finite distributive lattices to show that the orbit decomposition of $V(D)$ gives a lattice isomorphic to the lattice of contractions of the bounded poset of join-irreducibles $\widehat{P}$ of D. Then we describe the singular locus of $V(D)$ by applying some general theory of toric varieties to the fan dual to the order polytope of $P : V(D)$ is nonsingular along an orbit closure if and only if each

fibre of the corresponding contraction is a tree. Finally, we examine the local rings and associated graded rings of orbit closures in $V(D)$. This leads to a second (self-contained) proof that the singluar locus is as described, and a similar combinatorial criterion for the normal link of an orbit closure to be irreducible.

95-03 **Details Pertaining to "A Two-Line Algorithm for Proving Terminating Hypergeometric Identities"**

Lily Yen

95-04 **A Two-Line Algorithm for Proving Terminating Hypergeometric Identities**

Lily Yen

We give an algorithm that takes a proposed terminating hypergeometric identity of the form

$$\Sigma_k \; F(n,k) = f(n), \qquad n \geq n_0$$

and directly from $n_0, F(n,k)$ and $f(n)$ computes an explicit integer n_1 such that if the equality holds for all $n \; \epsilon \; \{n_0, n_0+1, \ldots, n_1\}$ then $\Sigma_k \; F(n,k) = f(n)$ for all $n \geq n_0$; otherwise the proposed identity is obviously false. This paper contains the first estimate for n_1.

95-05 **Group invariance and convex matrix analysis**

A. S. Lewis

Certain interesting classes of functions on a real inner product space are invariant under an associated group of orthogonal linear transformations. This invariance can be made explicit via a simple decomposition. For example, rotationally invariant functions on $\mathbb{R}^2$ are just even functions of the Euclidean norm, and functions on the Hermitian matrices (with trace inner product) which are invariant under unitary

similarity transformations are just symmetric functions of the eigenvalues. We develop a framework for answering geometric and analytic (both classical and nonsmooth) questions about such a function by answering the corresponding question for the (much simpler) function appearing in the decomposition. The aim is to understand and extend the foundations of eigenvalue optimization, matrix approximation, and semidefinite programming.

Keywords and phrases: convexity, group invariance, nonsmooth analysis, semidefinite program, eigenvalue optimization, Fenchel conjugate, subdifferential, spectral function, unitarily invariant norm, Schur convex, extreme point, von Neumann's lemma

95-06 **The Stockhausen problem and its generalizations**

Ronald C. Read and Lily Yen

We consider problems in the enumeration of sequences suggested by the problem of determining the number of ways of performing a piano composition (*Klavierstück XI*) by Karlheinz Stockhausen.

95-07 **Covers of Complete Graphs**

C. D. Godsil

We study antipodal distance-regular covers of complete graphs. The first part of the paper gives an introduction to the basic theory and the main constructions. In the second part, we turn to *linear* covers, which can be described geometrically.

95-08 **Maps in locally orientable surfaces, an integral representation, zonal polynomials, and the monopole series**

I.P. Goulden and D. M. Jackson

The genus series for maps is the generating series for the number of rooted maps with a given number of vertices and faces of each degree, and a given number of edges. It captures topological information

about surfaces, and appears in questions arising in statistical mechanics, topology, group rings, and certain aspects of free probability theory. An expression has been given previously for the genus series for maps in locally orientable surfaces in terms of zonal polynomials. The purpose of this paper is to derive an integral representation for the genus series. We then show how this can be used in conjunction with integration techniques to determine the genus series for monopoles in locally orientable surfaces. This complements the analogous result for monopoles in orientable surfaces previously obtained by Harer and Zagier.

95-09 **Stirling numbers for complex arguments**

Bruce Richmond and Donatella Merlini *

We define the Stirling numbers for complex values and obtain extensions of certain identities involving these numbers. We also show that the generalization is a natural one for proving unimodality and monotonicity results for these numbers. The definition is based on the Cauchy integral formula and can be used for many other combinatorial numbers.

95-10 **Raney paths and a combinatorial relationship between rooted nonseparable planar maps and two-stack-sortable permutations**

I.P. Goulden and J. West *

An encoding of the set of two-stack-sortable permutations ($\mathcal{TSS}$) in terms of lattice paths and ordered lists of strings is obtained. These lattice paths are called Raney paths. The encoding yields combinatorial decompositions for two complementary subsets of $\mathcal{TSS}$, which are the analogues of previously known decompositions for the set of nonseparable rooted planar maps ($\mathcal{NS}$). This provides a combinatorial relationship between $\mathcal{TSS}$ and $\mathcal{NS}$, and hence a bijection is determined between these sets that is different, simpler and more refined than the previously known bijection.

95-11 **An Interior-Point Method for Approximate Positive Semidefinite Completions**

Charles R. Johnson *, Brenda Kroschel * and Henry Wolkowicz

Given a nonnegative, symmetric, matrix of weights, H, we study the problem of finding an Hermitian, positive semidefinite matrix which is closest to a given Hermitian matrix, A, with respect to the weighting H. This extends the notion of exact matrix completion problems in that, $H_{ij} = 0$ corresponds to the element A_{ij} being *free* (unspecified), while H_{ij} large in absolute value corresponds to the element A_{ij} being *approximately fixed* (specified).

We present optimality conditions, duality theory, and two primal-dual interior-point algorithms. Because of sparsity considerations, the dual-step-first algorithm is more efficient for a large number of free elements, while the primal-step-first algorithm is more efficient for a large number of fixed elements.

Included are numerical tests that illustrate the efficiency and robustness of the algorithms.

Keywords and Phrases: positive definite completions, best nonnegative approximation, semi-definite programming, primal-dual interior point methods, complementarity problems.

95-12 **Strong Duality for Semidefinite Programming**

Motakuri Ramana*, Levent Tunçel and Henry Wolkowicz

It is well known that the duality theory for linear programming (LP) is powerful and elegant and lies behind algorithms such as simplex and interior-point methods. However, the standard Lagrangian for nonlinear programs requires constraint qualifications to avoid duality gaps.

Semidefinite linear programming (SDP) is a generalization of LP where

the nonnegativity constraints are replaced by a semidefiniteness constraint on the matrix variables. There are many applications, e.g. in systems and control theory and in combinatorial optimization. However, the Lagrangian dual for SDP can have a duality gap.

We discuss the relationships among various duals and give a unified treatment for strong duality in semidefinite programming. These duals guarantee strong duality, i.e. a zero duality gap and dual attainment. This paper is motivated by the recent paper by Ramana where one of these duals is introduced.

Keywords and Phrases: Semidefinite linear programming, strong duality, Löwner partial order, symmetric positive semidefinite matrices.

95-13 **Combining Semidefinite and Polyhedral Relaxations for Integer Programs**

C. Helmberg *, S. Poljak *, F. Rendl *, and H. Wolkowicz

We present a general framework for designing semidefinite relaxations for constrained 0-1 quadratic programming and show how valid inequalities of the cut-polytope can be used to strengthen these relaxations. As examples we improve the ϑ-function and give a semidefinite relaxation for the quadratic knapsack problem. The practical value of this approach is supported by numerical experiments which make use of the recent development of efficient interior point codes for semidefinite programming.

Keywords and phrases: integer linear programming, semidefinite programming, quadratic 0-1 optimization, interior point methods

95-14 **Characterizations of the Barrier Parameter of Homogeneous Convex Cones**

Osman Güler * and Levent Tunçel

We characterize the barrier parameter of the optimal self-concordant barriers for homogeneous cones. In particular, we prove that for homogeneous convex cones this parameter is the same as the rank of the corresponding Seigel domain. We also provide lower bounds on the barrier parameter in terms of the Carathéodory number of the cone. The bounds are tight for homogeneous self-dual cones.

Keywords and Phrases: Homogeneous cones, barrier functions, self-concordance, rank, Carathéodory number

95-15 **A Two-Line Algorithm for Proving q-Hypergeometric Identities**

Lily Yen

We show that q-hypergeometric identities $\Sigma_k F(n,k) = 1$ can be proved by checking that they are correct for only finitely many, N say, values of n. We give a specific *a priori* formula for N, as a polynomial of degree 24 in the parameters of $F(n,k)$. We see that because of the presence of "q", the estimates of N can be made smaller than the general estimates that were found in the author's thesis [8]. As an example of the method we show that the q-Vandermonde identity can be *proved* by "only" checking that its first 2358 cases (i.e., values of n) are correct, by direct computation.

95-16 **A Direct Combinatorial Proof for Stockhausen's Problem**

Lily Yen

We consider problems in the enumeration of sequences suggested by the problem of determining the number of ways of performing a piano composition (*Klavierstück XI*) by Karlheinz Stockhausen. An explicit formula and a direct combinatorial proof for the general problem are given.

Keywords and Phrases: sequences, bijections, sign-reversing involution, Gray codes

95-17 **A Symmetric Functions Approach to Stockhausen's Problem**

Lily Yen

We consider problems in sequence enumeration suggested by Stockhausen's problem. The analysis makes extensive use of techniques from the theory of symmetric functions. Each algebraic step is examined to derive information for formulating a direct combinatorial construction for such problems.

95-18 **Transitive factorisations into transpositions, and Lagrange inversion**

I. P. Goulden and D. M. Jackson

We determine the number of ordered factorisations of an arbitrary permutation on n symbols into transpositions such that the factorisations have minimal length and such that the factors generate the full symmetric group on n symbols. Such factorisations of the identity permutation have been considered by Crescimanno and Taylor in connection with a class of topologically distinct holomorphic maps on the sphere. As with the Macdonald's construction for symmetric functions that multiply as the classes of the class algebra, essential use is made of Lagrange inversion.

95-19 **Von Neumann's Lemma and a Chevalley-type theorem for convex functions on Cartan subspaces**

A. S. Lewis

We draw some striking connections between classical Lie theory and convex analysis. In particular, Von Neumann's characterization of unitarily invariant matrix norms is derived as a special case of a convex analogue of the Chevalley Restriction Theorem. The key tool is a generalization of Von Neumann's Lemma (an inequality relating the trace inner product of two matrices with their singular values), to the setting of a Cartan subspace of a real semisimple Lie algebra. Impor-

tant consequences include elegant and powerful Fenchel conjugacy and subgradient formulae, useful in eigenvalue optimization, semidefinite programming and matrix approximation.

Keywords and Phrases: convexity, Lie algebra, Von Neumann's Lemma, Cartan decomposition, Chevalley Restriction Theorem, semidefinite program, eigenvalue optimization, Fenchel conjugate, subdifferential, unitarily invariant norm

95-20 **On the interplay among entropy, variable metrics and potential functions in interior-point algorithms**

Levent Tunçel and Michael J. Todd *

We are motivated by the problem of constructing a primal-dual barrier function whose Hessian induces the (theoretically and practically) popular symmetric primal and dual scalings for linear programming problems. Although this goal is impossible to attain, we show that the primal-dual entropy function may provide a satisfactory alternative. We study primal-dual interior-point algorithms whose search directions are obtained from a potential function based on this primal-dual entropy barrier. We provide polynomial iteration bounds for these interior-point algorithms. Then we illustrate the connections between the barrier function and a reparametrization of the central path equations. Finally, we consider the possible effects of more general reparametrizations on infeasible-interior-point algorithms.

Keywords and Phrases: linear programming, interior-point algorithms, primal-dual, entropy, potential function

95-21 **A New Finite Pivot Rule for the Simplex Method**

Charles E. Haff

A new finite version of the simplex method is given under a pivot

rule which, in contrast to lexicography and lowest-index rules, is independent of data index ordering and is thereby faithful to the Bourbaki concept of matrix.

95-22 **Eigenvalue-constrained faces**

A.S. Lewis

We characterize the exposed faces of convex sets $\mathcal{C}$ of symmetric matrices, invariant under orthogonal similarity ($U^T\mathcal{C}U = \mathcal{C}$ for all orthogonal U). Such sets $\mathcal{C}$ are exactly those determined by eigenvalue constraints: typical examples are the positive semidefinite cone, and unit balls of common matrix norms. The set $\mathcal{D}$ of all diagonal matrices in $\mathcal{C}$ is convex if and only if $\mathcal{C}$ is, and $\mathcal{D}$ is invariant under the group of permutations (acting on diagonal entries). We show how any exposed face of $\mathcal{C}$ is naturally associated with an exposed face of $\mathcal{D}$, by relating the stablizer groups of the two faces.

Keywords and Phrases: exposed face, semidefinite cone, eigenvalue optimization, unitarily invariant norm

95-23 **Nested Chain Partitions of Hamiltonian Filters**

David G. C. Horrocks

Let P be a poset, consisting of all sets $X \subseteq [n] = \{1, 2, \ldots, n\}$ which contain at least one of a given collection $\mathcal{F}$ of 2-subsets of $[n]$, ordered by inclusion. By modifying a construction of Greene and Kleitman, we show that if $\mathcal{F}$ is hamiltonian, that is, contains $\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}$ and $\{1, n\}$, then P is a nested chain order. We examine the Sperner-type properties of such posets, and provide further support for a conjecture of Lih.

95-24 **Connexion coefficients for the symmetric group, free products in operator algebras and random matrices**

I. P. Goulden and D. M. Jackson

There are some significant points of contact between the principal topics

of the Workshop and some fundamental structures and algebraic ideas that are used in algebraic combinatorics. The combinatorial structures are beguilingly simple: the product structure of permutations, and two-cell embeddings of graphs in closed surfaces (orientable and locally orientable) without boundaries. We show how these are related to mixed moments to two free random variables, and to integrals that involve random Hermitian complex, and real symmetric matrices.

Character theoretic properties of the Schur functions and the zonal polynomials are heavily exploited. Their joint generalisation, the Jack symmetric function not only has combinatorial significance in this context, but also is associated with generalisations of the Selberg-type integrals that extend the Hermitian complex and real symmetric cases.

In this paper we describe the relationship between these various ideas, and describe briefly some of the significant consequences of this methodology to other questions.

95-25 **Global and Local Quadratic Minimization**

M. J. Best and B. Ding

We present a method which when applied to certain non-convex QP will locate the global minimum, *all* isolated local minima and some of the non-isolated local minima. The method proceeds by formulating a (multi) parametric convex QP in terms of the data of the given non-convex QP. Based on the solution of the parametric QP, an unconstrained minimization problem is formulated. This problem is piece-wise quadratic. A key result is that the isolated local minimizers (including the global minimizer) of the original non-convex problem are in one-to-one correspondence with those of the derived unconstrained problem.

The theory is illustrated with several numerical examples. A numerical procedure is developed for a special class of non-convex QP's. It

is applied to a problem from the literature and verifies a known global optimum and in addition, locates a previously unknown local minimum.

Keywords and Phrases: Global optimization, parametric quadratic programming, non-convex quadratic program.

95-26 **Approximation algorithms for feasible cut and multicut problems**

Bo Yu and Joseph Cheriyan

Let $G = (V, E)$ be an undirected graph with a capacity function $u : E \to \Re_+$ and let $S_1, S_2, \ldots, S_k$ be k commodities, where each S_i consists of a pair of nodes. A set X of nodes is called feasible if it contains no S_i, and a cut $(X, \overline{X})$ is called feasible if X is feasible. Several optimization problems on feasible cuts are shown to be NP-hard. A 2-approximation algorithm for the minimum-capacity feasible v^*-cut problem is presented. The multicut problem is to find a set of edges $F \subseteq E$ of minimum capacity such that no connected component of $G \backslash F$ contains a commodity S_i. It is shown that an α-approximation algorithm for the minimum-ratio feasible cut problem gives a $2\alpha(1 + \ln T)$-approximation algorithm for the multicut problem, where T denotes the cardinality of $\bigcup_i S_i$. A new approximation guarantee of $0(t \log T)$ for the minimum capacity-to-demand ratio Steiner cut problem is presented; here each commodity S_i is a set of two or more nodes and t denotes the maximum cardinality of a commodity S_i.

Keywords and Phrases: feasible cut, multicut, flow, set covering problem, NP-hard problem, approximation algorithm

95-27 To be submitted

95-28 **Multivariable Lagrange inversion, Gessel-Viennot cancellation and the Matrix Tree Theorem**

I. P. Goulden and D. M. Kulkarni*

A new form of multivariable Lagrange inversion is given, with determinants occurring on both sides of the equality. These determinants are principal minors, for complementary subsets of row and column indices, of two determinants that arise singly in the best known forms of multivariable Lagrange inversion. A combinatorial proof is given by considering functional digraphs, in which one of the principal minors is interpreted as a Matrix Tree determinant, and the other by a form of Gessel-Viennot cancellation.

Addresses

Osman Güler
Department of Mathematics and Statistics
University of Maryland Baltimore County
Baltimore, Maryland 21228
U.S.A.
e-mail: guler@math.umbc.edu

C. Helmberg
Technische Universität Graz
Institut für Mathematik
Kopernikusgasse 24
A-8010 Graz
Austria

Charles R. Johnson
College of William & Mary
Department of Mathematics
Williamsburg, Virginia 23187-8795
U.S.A.
e-mail crjohnso@cs.wm.edu

Brenda Kroschel
College of William & Mary
Department of Mathematics
Williamsburg, Virginia 23187-8795
U.S.A.
e-mail kroschel@cs.wm.edu

Devadatta Kulkarni
Oakland University
Rochester, MI 48309-4401
U.S.A.
e-mail kulkarni@vela.acs.oakland.edu

Donatella Merlini
Dipartimento di Sistemi e Informatica
Via Lombroso 6/17 - Firenze (Italy)
e-mail dada@cellini.ing.unifi.it

S. Poljak
Universität Passau
Institut für Mathematik und Informatik
InnstraBe 33
94030 Passau, Germany

Motakuri Ramana
Department of Industrial and Systems Engineering
University of Florida
Gainesville, FL 32611
U.S.A.
e-mail ramana@iseipc.ise.ufl.edu

F. Rendl
Technische Universität Graz
Institut für Mathematik
Kopernikusgasse 24
A-8010 Graz, Austria

Michael J. Todd
School of Operations Research and Industrial Engineering
Cornell University
Ithaca, NY 14853-3801
U.S.A.

J. West
CECM
Department of Mathematics and Statistics
Simon Fraser University
Burnaby, BC V5A 1S6
Canada