Discrete Optimization
Comprehensive Examination
1 p.m. to 4 p.m., Tuesday, June 6, 2005
Examiners: Bill Cunningham and Jim Geelen

Note. The numbers on the right are approximate values of questions. They total 100. The presentation of your solutions is important in assigning marks. Do not use results without proof unless the question allows it.

1. Let $G = (V, E)$ be a directed graph with arc-weights $c \in \mathbf{Z}^E$ and capacities $u \in \mathbf{Z}_+^E$. Now let $\mathcal{C}$ denote the set of directed circuits of G. Consider the following linear programs: [23]

$$(P_1)\begin{cases} \text{Maximize} & \sum_{e\in E} c_e x_e \\ \text{Subject to} & f_x(v) = 0 \quad v \in V \\ & 0 \le x_e \le u_e \quad e \in E \end{cases}$$

$$(P_2)\begin{cases} \text{Maximize} & \sum_{C\in\mathcal{C}} c(C) z_c \\ \text{Subject to} & \sum(z_C : e \in C \in \mathcal{C}) \le u_e \quad e \in E \\ & z_C \ge 0 \quad C \in \mathcal{C} \end{cases}$$

   (a) Let x^* be an optimal solution to the integer programming problem associated to (P_1). Prove that x^* is an optimal solution of (P_1). You may not use results on flows, but you may use results on shortest paths.

   (b) Prove that (P_1) and (P_2) have the same optimal value. You may use standard results on flows.

   (c) Prove that (P_2) and its dual have optimal solutions that are integer-valued. You may use any results, other than this one.

2. (a) Let $G = (V, E)$ be a 3-edge-connected graph in which each node has degree exactly 3, and let $e \in E$. Prove that G has a perfect matching containing e. You may use results other than this one. [22]

   (b) Let $G = (V, E)$ be a graph and let c_e be an edge weight for each $e \in E$. Let $f(k)$ denote the maximum weight of a matching of size k. Briefly indicate how $f(k)$ could be computed efficiently. Prove, assuming $f(k)$, $f(k-1)$ and $f(k+1)$ are defined, that $f(k) \ge \min(f(k-1), f(k+1))$. You may use basic results on matching.

3. (a) State and prove the Matroid Intersection Theorem. You may use basic results on minors and submodularity. [20]

   (b) Let $M_1, \ldots, M_k$ be matroids on ground set S. Prove that there exists an independent set J_i of M_i, for $i = 1, \ldots, k$, such that $S = \cup_{i=1}^k J_i$, if and only if for every subset A of S we have $\sum_{i=1}^k r_i(A) \ge |A|$. You may use the Matroid Intersection Theorem, as well as basic matroid constructions.

4. (a) Let $Ax \leq b$ be a system of inequalities that has no solution. If A is m by n, prove that $Ax \leq b$ has a subsystem consisting of at most $n+1$ inequalities that also has no solution. You may use basic results from linear programming. [20]

   (b) Let $P = \{x \in \mathbf{R}^n : \ Ax \leq b\}$ be a rational polyhedron. Define the term *Chvátal-Gomory cut* for P. Let P' denote the set of points $x \in \mathbf{R}^n$ such that x satisfies all Chvátal-Gomory cuts for P. Prove that P' is itself a polyhedron.

5. For each of the following problems, indicate whether it admits a polynomial-time algorithm, or is NP-hard. If the former, indicate why, using known results. If the latter, prove it by reduction to a known NP-hard problem. (A list of such problems is attached.) [15]

   (a) Given a graph G, does G have 2 edge-disjoint spanning trees?

   (b) Given a graph G, does G have 2 disjoint perfect matchings?

   (c) Given a graph $G = (V, E)$, $c \in \mathbb{Z}^E$ and an integer k, does there exist $x \in \mathbf{R}^E$ such that $0 \leq x \leq 1$, $x(T) \geq 1$ whenever T is the edge-set of a spanning tree of G, and $c^T x \leq k$?

NP-complete problems

Disjoint Tree and Path

INSTANCE: Graph G, nodes r, s of G.
QUESTION: Does G have an (r, s)-path and a spanning tree that are edge-disjoint?

3-Edge Colouring

INSTANCE: Three-regular graph $G = (V, E)$.
QUESTION: Can each edge be assigned one of 3 different colours so that edges incident to the same node get distinct colours?

Degree-constrained Spanning Tree

INSTANCE: Graph $G = (V, E)$ and integer k.
QUESTION: Is there a spanning tree T of G such that every node of T has degree at most k?

Max Cut

INSTANCE: Graph $G = (V, E)$ and integer k.
QUESTION: Does there exist a cut C of G with $|C| \geq k$?