Discrete Optimization Comprehensive Exam — Spring 2016

Examiners: Joseph Cheriyan and Bertrand Guenin

Instructions: All four questions have the same number of marks. If you are pressed for time it is better to answer some questions completely than to give partial answers to all questions.

Unless otherwise stated, do not use results without proofs. If you are asked to state or prove a result in a part of a question, you may use it without proof in subsequent parts of the question. The reference CCPS refers to the Cook-Cunningham-Pulleyblank-Schrijver book.

$\mathcal{P}$ denotes the class of problems solvable in polynomial time (see #3(d), #4(c)).

Feel free to ask the proctors if any notation is unclear.

Problem 1: Network Flows **(20 marks)**

In this question, you may use the max-flow min-cut theorem without proof.

(a) Let $G = (V, E)$ be a digraph, let $u \in \mathbb{R}^E_+$ assign a nonnegative capacity to each arc, and let $b \in \mathbb{R}^V$ assign a demand to each node. State and prove necessary and sufficient conditions for the existence of a flow $x \in \mathbb{R}^E$ such that

$$\begin{aligned} f_x(v) &= b_v, && \text{for all } v \in V \\ 0 \le x_e &\le u_e && \text{for all } e \in E. \end{aligned}$$

(Recall that for $x \in \mathbb{R}^E$, $f_x(v)$ means $\sum_{e\in\delta(V\setminus v)} x_e - \sum_{e\in\delta(v)} x_e$.)

(b) Let $G = (V, E)$ be an undirected simple graph, and let $h : V \to \mathbb{Z}_+$ assign a non-negative integer to each node. An *h-factor* means a subgraph H of G such that each node v is incident to $h(v)$ edges of H.

Suppose that G is a bipartite graph with node bipartition L, R.
Prove: G has an h-factor if and only if

i. $h(L) = h(R)$, and
ii. $\forall A \subseteq L, \forall B \subseteq R :\ h(A) \le h(R-B) + m(A,\, B)$, where $m(A, B)$ denotes the number of edges of G that have one end-node in A and the other end-node in B.

Problem 2: Matroids **(20 marks)**

(a) State the matroid intersection theorem, and prove that the maximum is at most the minimum.

(b) Let M be a matroid, and let N be obtained from M by contracting a subset of the elements J. Express the rank function of N in terms of the rank function of M. No need to justify your answer.

(c) Let $M = (S, \mathcal{I})$ be a matroid with rank function r. Suppose that $|S| = 2k$ and S is partitioned into k pairs, and moreover, we fix an ordering on each of the pairs; thus, we have ordered pairs $(b_1, g_1), (b_2, g_2), \ldots, (b_k, g_k)$, and we have $S = S_1 \cup S_2$, where $S_1 = \{b_1, \ldots, b_k\}$, and $S_2 = \{g_1, \ldots, g_k\}$. (Informally, we have k pairs, each consists of a "boy" and a "girl", and we have a matroid M on the $2k$ elements.) By a *nice independent* set $J \subseteq S$ we mean that J is an independent set of M and, moreover, $b_i \in J \implies g_i \in J$, $\forall i = 1, \ldots, k$. (Thus, if a "boy" b_i is in a nice independent set, then the "girl" g_i cannot be left out of the set.) We need some notation: for $A \subseteq S_1$, let $\widehat{A} = \{g_i \mid b_i \in A\}$; observe that $\widehat{S_1} = S_2$.
Prove:

$$\max\{|J| \ :\ J \text{ is a nice independent set}\} \quad = \quad \min\{r(S - A) + r(\widehat{A}) \ :\ A \subseteq S_1\}.$$

HINT: Matroid intersection theorem, with groundset S_1. We have two matroids M_1 and M_2. $A \subseteq S_1$ is an independent set of M_1 if $\widehat{A}$ is an independent set of M.

Problem 3: Matchings **(20 marks)**

(a) State the Tutte-Berge formula.

A graph is *hypomatchable* if for every node v there exists a matching that misses only v. Consider a graph G that has no perfect matching, and let B denote a maximum cardinality minimizer in the Tutte-Berge formula (thus, $|B|$ is maximum over all the sets that achieve the minimum in the formula).

(b) Show that every even component of $G \setminus B$ has a perfect matching.

(c) Show that every odd component of $G \setminus B$ is hypomatchable.

(d) (This part is independent of the previous parts.) Suppose that we are given a graph H with a matching N. Indicate for each of the following decision problems if it is in $\mathcal{P}$, or if it is unlikely to be in $\mathcal{P}$ ($\mathcal{NP}$-complete?). Justify your answers in brief.

   i. Does H have an N-augmenting path?

   ii. Does H have an N-alternating path that uses every edge of N?

Problem 4: Polyhedral theory **(20 marks)**

An $n \times n$ $\{0,1\}$ matrix M is an *odd hole* if

i. n is odd,

ii. M has exactly two 1s in each row and each column,

iii. no proper submatrix of M satisfies ii.

For instance the following is an odd hole,

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Given a $\{0,1\}$ matrix A, the *set partition polytope* is the polytope:

$$P(A) = \{x | Ax = \mathbf{1}, \mathbf{0} \leq x \leq \mathbf{1}\}$$

where $\mathbf{0}$ (resp. $\mathbf{1}$) is the vector of all zeros (resp. ones).

(a) Show that if A is an odd hole then $P(A)$ is fractional.

(b) Show that if $P(A)$ is fractional then A has a submatrix B where each row has exactly two 1s.

   HINTS:

   1. Pick a smallest[1] submatrix B of A such that $P(B)$ is fractional.
   2. Show that $P(B)$ has an extreme point $\bar{x}$ where $0 < \bar{x}_i < 1$, for every i and show that B is square.
   3. Let j be a row index of B and let B' be obtained from B by removing row j.
   4. Show that $P(B')$ has exactly two integer points, y, z and that $\bar{x}$ is a convex combination of y, z.

(c) (This part is independent of the previous parts.) Suppose that we are given a $\{0,1\}$ matrix A and an inequality $w^\top x \leq t$ where w and t are rational. Indicate for each of the following decision problems if it is in $\mathcal{P}$ or if it is unlikely to be in $\mathcal{P}$ ($\mathcal{NP}$-complete?). Justify your answers in brief.

   i. Is $w^\top x \leq t$ valid for $P(A)$?

   ii. Is $w^\top x = t$ a supporting hyperplane for $P(A)$?