Problem 1: Network Flows (20 marks)

a) State The Max-Flow Min-Cut (MFMC) Theorem.

b) Using the MFMC Theorem, find a sufficient and necessary condition for the existence of a circulation in a directed graph $G = (V, E)$ with lower bounds $\ell$ and upper bounds $u$, i.e. a solution to,

\[
\begin{align*}
\sum_{e \in E} x_e &= 0 \quad (u \in V) \\
\ell_e &\leq x_e \leq u_e \quad (e \in E)
\end{align*}
\]

HINT: Define $x'_e = x_e - \ell_e$ and $u'_e - \ell_e$ for all $e \in E$.

Problem 2: Matching polytope, blossom algorithm (20 marks)

Let $G = (V, E)$ be a graph with non-negative edge weights $c = (c_e : e \in E)$. Edmonds showed that the maximum weight of a matching of $G$ is equal to the optimal value of the following linear-program (P).

\[
\begin{align*}
\max \sum_{e \in E} (c_e x_e) \\
\text{subject to} \\
x(\delta(v)) &\leq 1 \quad (v \in V) \\
x(\gamma(S)) &\leq (|S| - 1)/2 \quad (S \subseteq V, |S| \geq 3, |S| \text{ odd}) \\
x_e &\geq 0, \quad (e \in E)
\end{align*}
\]

a) Write the linear-programming dual (D) of (P).

b) Suppose that $c_e$ is an integer for all $e \in E$. Then the blossom algorithm shows that (D) has an optimal solution such that the variables corresponding to constraints (1) are half-integer valued and the variables corresponding to constraints (2) are integer valued. Show how to modify such a half-integer optimal solution to (D) to obtain an integer optimal solution to (D).

HINT: An optimal solution to (D) has an even number of variables that have non-integer values.

c) Using a result on Total Dual Integrality deduce that the polyhedron described by (1) – (3) is integral.
Problem 3: Matroid intersection  

(20 marks)

a) State the Matroid Intersection Theorem for a pair of matroids $M_1$ and $M_2$.

b) Prove that the max is at most equals to the min.

c) Simplify the statement for the case where $M_2$ is the dual of $M_1$.

d) Given a connected planar graph $G$ and a non-negative integer $k$, find a necessary and sufficient for the existence of $k$ edges of $G$ such that its removal keeps $G$ and the planar dual of $G$ connected.

e) Is the previous problem solvable in polynomial time? Justify your answer.

Problem 4: Complexity  

(15 marks)

a) Define the problem classes $\mathcal{NP}$, co-$\mathcal{NP}$, and $\mathcal{NP}$-complete.

b) Consider the following two problems.

- **Directed Hamiltonian Circuit:**
  Given a directed graph, does it have a directed Hamiltonian circuit?

- **Undirected Hamiltonian Circuit:**
  Given an undirected graph, does it have a Hamiltonian circuit?

Use the fact that Directed Hamiltonian Circuit is $\mathcal{NP}$-complete to show that Undirected Hamiltonian Circuit is also $\mathcal{NP}$-complete.