Answer as many questions as time permits. Each question part is worth 10 points (total: 90 points).

1. An \textit{\(R\)-sequence} is a sequence \((c_1, \ldots, c_n)\) of integers \(c_i \geq -1\) such that \(c_1 + \cdots + c_n = 0\) and for all \(1 \leq i < n\), \(c_1 + \cdots + c_i \geq 0\). For each \(n \in \mathbb{N}\), determine the number of \(R\)-sequences of length \(n\).

2. Let \(\phi : X \to X\) be an endofunction. A vertex \(v \in X\) is \textit{recurrent} if there is some positive integer \(k \geq 1\) such that \(\phi^k(v) = v\). Let \(Q\) be the species (class) of endofunctions \(\phi : X \to X\) such that if \(v \in X\) has \(|\phi^{-1}(v)| \geq 2\), then \(v\) is recurrent. Obtain a formula for the exponential generating function

\[
Q(x) = \sum_{n=0}^{\infty} \frac{|Q_n| x^n}{n!}.
\]

(Note: \(Q_n\) denotes the set of all such endofunctions when \(X = \{1, 2, \ldots, n\}\).)

3. Let \(\mathcal{R}\) be the species (class) of rooted labelled trees (RLTs). Let \(c_k(T,v)\) denote the number of nodes with exactly \(k \in \mathbb{N}\) children in a RLT \((T,v)\). Let \(\overline{c}_k(n)\) denote the average value of \(c_k(T,v)\) among all \(n^{n-1}\) RLTs on the set \(\{1, 2, \ldots, n\}\).

(a) Obtain an expression which determines the exponential generating function

\[
R(x, y) = \sum_{n=0}^{\infty} \left( \sum_{(T,v) \in \mathcal{R}_n} y^{c_k(T,v)} \right) \frac{x^n}{n!}.
\]

(b) Show that for \(0 \leq k \leq n\),

\[
\overline{c}_k(n) = \frac{n^2}{(n-1)^{k+1}} \binom{n-1}{k} \left(1 - \frac{1}{n}\right)^n.
\]
4. For \( n, k \in \mathbb{N} \), let \( a_{n,k} \) denote the number of lattice paths starting at \((0, 0)\) and ending at \((n, n)\), such that all points \((x, y)\) on the path satisfy \( x \leq y \leq x + k \).

Let \( A_k(t) = \sum_{n \geq 0} a_{n,k} t^n \). Find a recurrence relation for \( A_k(t), k \geq 1 \). Hence, or otherwise, prove that

\[
A_k(t) = \frac{\sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^m {k-m \choose m} t^m}{\sum_{m=0}^{\lfloor \frac{k+1}{2} \rfloor} (-1)^m {k+1-m \choose m} t^m}.
\]

5. For \( m, n, k \in \mathbb{N} \), let \( a(m, n, k) \) denote the number of \( m \)-by-\( n \) matrices with entries in \( \{0, 1\} \), with no rows or columns consisting entirely of 0s, and with exactly \( k \) 1s. Consider the generating series

\[
A(x, y, z) = \sum_{m,n,k \in \mathbb{N}} a(m, n, k) x^m y^n z^k.
\]

(a) Explain why

\[
A(x, y, z) = \exp(-x-y) \sum_{m,n \in \mathbb{N}} (1+z)^{mn} \frac{x^m y^n}{m! n!}.
\]

(b) From part (a) or otherwise, derive a partial differential equation satisfied by \( A(x, y, z) \), and show that \( \{a(m, n, k) : m, n, k \in \mathbb{N}\} \) satisfies a linear recurrence relation.

6. Let \( S_n \) denote the set of permutations of \( \{1, 2, \ldots, n\} \). A permutation \( \sigma \in S_n \) has a descent at position \( i \) if \( \sigma(i) > \sigma(i+1) \).

(a) Let \( 1 \leq a_1 < a_2 < \cdots < a_m \leq n - 1 \) be integers. Use Inclusion/Exclusion to show that the number of permutations in \( S_n \) with descents at the positions \( a_1, a_2, \ldots, a_m \) and nowhere else is

\[
n! \sum_{j=0}^{m} (-1)^{m-j} \sum_{1 \leq i_1 < i_2 < \cdots < i_j = m} \prod_{\ell=1}^{j+1} \frac{1}{(a_{i_\ell} - a_{i_{\ell-1}})!}.
\]

In the product, we make the convention that \( a_{i_0} = 0 \) and \( a_{i_{j+1}} = n \).

(b) Let \( A = (A_{ij})_{i,j=0,\ldots,m} \) be the matrix

\[
A_{ij} = \begin{cases} 
\frac{1}{(a_{j+1} - a_i)!} & \text{if } j + 1 \geq i \\
0 & \text{otherwise},
\end{cases}
\]

where \( a_0 = 0 \) and \( a_{m+1} = n \). Prove that the summation in part (a) is equal to \( n! \det(A) \).