Graph Theory Comprehensive Examination 2004

Total: 70 marks

1. Prove Ramsey's theorem in the case $k = 2$. In other words, show that for positive integers r, p_1 and p_2, there is a finite positive integer N such that every 2-colouring of the r-subsets of an N-element set has an i-homogeneous p_i-set (i.e. a p_i-set whose r-sets all have colour i) for either $i = 1$ or $i = 2$. [10 marks]

2. By using Euler's formula or otherwise, prove that every simple planar graph with n vertices but no cycle of length 3 or 4 has at most $5(n-2)/3$ edges. [5 marks]

3. (a) Give a self-contained proof of Menger's theorem that if G is a graph and A and B are subsets of the vertex set of G then the minimum number of vertices separating A from B is equal to the maximum number of disjoint $A - B$ paths in G. [10 marks]

   (b) Show how the above version of Menger's theorem implies the following. For any two non-adjacent vertices u and v of a graph G, the minimum number of vertices (not u or v) whose removal separates u and v is equal to the maximum number of paths from u to v which intersect pairwise only at u and v. [5 marks]

4. (a) Prove that every tournament is semi-Hamiltonian, that is, has a path which passes through every vertex. [5 marks]

   (b) Show that every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ has a Hamilton cycle. [10 marks]

5. (a) Let G be a graph. Show that the number of flows on G over $\mathbb{Z}_n$ is a polynomial in n. [5 marks]

   (b) Prove that a graph G is 2^k-colourable if and only if it is the union of k bipartite graphs. [5 marks]

6. (a) Show that $L(K_n)$, the line graph of K_n, is strongly regular and determine its eigenvalues. [10 marks]

   (b) What are the multiplicities of the eigenvalues of $L(K_{10})$? [5 marks]