GRAPH THEORY COMPREHENSIVE EXAMINATION 2005

Total: 78 marks

If you wish to use the result of a well-known theorem, state clearly the theorem you are using. If you are in any doubt about what needs to be stated, please ask an examiner.

1. (a) Let $\lambda_1(H)$ denote the largest eigenvalue of the adjacency matrix of a graph H. Show that if F is an induced subgraph of a graph G then $\lambda_1(F) \leq \lambda_1(G)$. [6 marks]

   (b) Prove that if G is a graph with n vertices and B is the incidence matrix of G, then the rank of B plus the number of bipartite components of G is exactly equal to n. [6 marks]

2. Show that every 3-connected graph with no K_5 or $K_{3,3}$ minor is planar. [12 marks]

   (Of course you may not assume Kuratowski's theorem, but you may assume that

   (i) every 3-connected graph G with at least 5 vertices has an edge e such that contracting e produces a 3-connected graph;

   (ii) every face of a 2-connected plane graph is bounded by a cycle.)

   Hint: you may contract a certain edge xy, and embed the resulting graph in the plane by induction.

3. (a) Prove Vizing's Theorem: Every graph G has an edge colouring with at most $\Delta(G)+1$ colours, where Δ denotes the maximum degree. [12 marks]

   (b) Prove that if G is a bipartite graph then it has an edge colouring with $\Delta(G)$ colours. [6 marks]

4. Let $T_{r-1}(n)$ denote the Turán graph, i.e. the $(r-1)$-partite graph with n vertices whose partition classes are as equal as possible in cardinality. Prove Turán's Theorem: Let G be a graph with n vertices that does not contain a copy of the complete graph K_r as a subgraph. Then $|E(G)| \leq |E(T_{r-1}(n))|$, and if equality holds then G is isomorphic to $T_{r-1}(n)$. [10 marks]

5. Let $k \geq 2$ and let G be a k-connected graph that does not have a Hamilton cycle. Prove that G has an independent set of size $k+1$. [8 marks]

6. (a) By following the steps below or otherwise, prove that a multigraph G has a k-flow if and only if it has a $\mathbb{Z}_k$-flow.

   (i) Let G have vertex set V and edge set E, and let $\vec{E} = \{(e, x, y) : e \in E; x, y \in V; e = xy\}$. Let g be a $\mathbb{Z}_k$-flow on G. Let F be the set of all functions $f : \vec{E} \to \mathbb{Z}$ satisfying

   - $f(e, x, y) = -f(e, y, x)$ for all $(e, x, y) \in \vec{E}$, $x \neq y$,
   - $|f(\vec{e})| < k$ for all $\vec{e} \in \vec{E}$,
   - $\sigma_k \circ f = g$ where σ_k is the natural map from $\mathbb{Z}$ to $\mathbb{Z}_k$.

   Show that $F \neq \emptyset$.

   (ii) For $f \in F$ define $K(f) = \sum_{x \in V} |f(x, V)|$, where

   $$f(x, V) = \sum_{\substack{\vec{e}=(e,x,y)\in\vec{E} \\ \text{with } x \neq y}} f(\vec{e}).$$

   Suppose $K(f) > 0$ for all $f \in F$. Prove that there exist vertices x and x' in the same component of G such that $f(x, V) > 0$ and $f(x', V) < 0$.

   (iii) Prove that there exists $f \in F$ with $K(f) = 0$, and conclude that G has a k-flow. Thus complete the proof of the theorem.

   [12 marks]

   (b) Prove that a cubic graph G has a 4-flow if and only if it is 3-edge-colourable. [6 marks]