This examination has two parts. A complete paper consists of all four questions from Part A and any two of the three from Part B.

**PART A: attempt all four questions**

**Question 1.** Let $G$ be a simple, connected graph with maximum degree $\Delta(G)$. Prove Brooks’ Theorem: if $\Delta(G) \geq 3$, then either $G$ is isomorphic to $K_{\Delta(G)+1}$ or $\chi(G) \leq \Delta(G)$.

**Question 2.** State and prove Turán’s Theorem on the number of edges in a simple graph not having the complete graph $K_t$ as a subgraph.

**Question 3.** (i) State and prove Hall’s Theorem on matchings in a bipartite graph.

(ii) Using Hall’s Theorem, prove the following version of Petersen’s 2-factor Theorem. Let $r$ be a positive integer. If $G$ is a $2r$-regular simple graph, then $G$ has a $2$-regular subgraph.

*Hint. You may use, without proof, Euler’s Theorem that there is a closed Euler tour if and only if every vertex has even degree.*

**Question 4.** Let $G$ be a connected simple graph with maximum degree $\Delta$ and adjacency matrix $A$. Prove each of the following.

(i) $\Delta$ is an eigenvalue of $A$ if and only if $G$ is $\Delta$-regular;

(ii) If $\Delta$ is an eigenvalue of $A$, then its multiplicity is 1.

(iii) Suppose $G$ is $\Delta$-regular and there exist integers $a \geq 0$ and $b \geq 1$ so that

- every pair of adjacent vertices have exactly $a$ common neighbours, and
- every pair of non-adjacent vertices have exactly $b$ common neighbours.

(a) Prove that $A^2 + (b-a)A + (b-\Delta)I = cJ$, where, respectively, $I$ and $J$ denote the identity and all-1’s matrices of the appropriate sizes.

(b) Using (iii) or otherwise, obtain a formula satisfied by all eigenvalues of $A$ other than $\Delta$.

... over for Part B
PART B: attempt any two of three

Question 5. Let $G$ be a graph that has either $K_5$ or $K_{3,3}$ as a (contraction and deletion) minor. Prove that $G$ has a subgraph that is a topological minor of either $K_5$ or $K_{3,3}$. (In a topological minor, an edge may be replaced by a path, and all such paths must be internally disjoint.)

Question 6. Let $G$ be a 3-connected simple graph having at least 5 vertices. Prove that $G$ has an edge $e$ so that $G/e$ (the graph obtained from $G$ by contracting $e$) is 3-connected.

Remark. We remove parallel edges in obtaining $G/e$.

Question 7. Let $G$ be a simple graph, with chromatic number $\chi(G)$, and let $k$ be a positive integer. Prove:

$\chi(G) \leq k$ if and only if there is an orientation $D$ of $G$ so that every directed path in $D$ has at most $k$ vertices.

(Hint. To show the existence of $D$ implies $\chi(G) \leq k$, let $E_0$ be a minimal set of edges so that $D - E_0$ has no directed cycles. Define $c(v)$ to be the number of vertices in a longest directed path in $D - E_0$ beginning at $v$ and show that $c$ defines a colouring of $G$ with at most $k$ colours.)