GRAPH THEORY COMPREHENSIVE EXAMINATION
SUMMER 2012
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Instructions. Attempt all six questions. You proofs should be self-contained.

1. A tournament is *transitive* if it has no directed cycles. Equivalently, the vertices can be ordered $v_1, \ldots, v_n$ such that every directed edge is of the form (v_i, v_j) with $i < j$. Prove that for all positive integers k, there is an integer N such that if D is a directed graph on $n \geq N$ vertices whose directed edges have been coloured red and blue, then D contains either an independent set of size k or a k-vertex transitive subtournament with all edges of the same colour.

2. Let X and Y be sets of vertices in a directed graph G. An (X, Y)-*path* is a directed path from a vertex in X to a vertex in Y. An (X, Y)-*separator* is a set of vertices that meets each (X, Y)-path. Prove that the maximum number of vertex disjoint (X, Y)-paths is equal to the minimum size of an (X, Y)-separator.

3 (a) Let Γ_1 and Γ_2 be finite abelian groups with $|\Gamma_1| = |\Gamma_2|$. Prove that, for any graph G, the number of nowhere-zero flows for Γ_1 is the same as the number of nowhere-zero flows for Γ_2.

(b) Let G be a graph with two edge-disjoint spanning trees. Prove that G has a nowhere zero $\mathbf{Z}_4$-flow.

4. Prove Dirac's theorem: a (simple) graph with $n \geq 3$ vertices and minimum degree at least $n/2$ has a Hamilton cycle.

5 (a) Prove that if a graph is edge-transitive but not vertex-transitive, and has no isolated vertices, then it is bipartite.

(b) Prove that if a graph is vertex- and edge-transitive, but not arc-transitive, then all its vertices have even degree.

6. A graph H is a *topological minor* of a graph G if there is a subdivision of H that is isomorphic to a subgraph of G. Let $\mathcal{G}$ be a class of graphs that is closed under taking minors and that has a finite set of excluded minors. Prove that $\mathcal{G}$ has a finite set of excluded topological minors.