C&O Comprehensive
Graph Theory

July 4 2017, 1-4PM

Examiners: Penny Haxell and Peter Nelson

Instructions: A complete examination consists of solutions to all six questions. All questions have equal value and, in multipart questions, all parts have equal value. All proofs should be derived from first principles unless indicated otherwise. All graphs are simple.

1. Prove that if G is a 3-connected graph with $|V(G)| \geq 5$, then there is an edge e of G for which G/e is 3-connected.

2. Prove that if G is a 3-connected graph with no K_5-minor or $K_{3,3}$-minor, then G is planar. You may use without proof the statement of Question 1, as well as the fact that, in a 2-connected plane graph, the boundary of each face is a cycle.

3. Prove Vizing's theorem: every graph G has an edge-colouring with $\Delta(G) + 1$ colours.

4. (a) Prove Tutte's theorem: a graph G has a perfect matching if and only if, for each $X \subseteq V(G)$, the graph $G - X$ has at most $|X|$ components of odd size.

   (b) Prove that every cubic graph with no cut edge has a perfect matching.

5. For each positive integer s, let $R^{\text{bp}}(s)$ denote the minimum integer n such that every colouring of the edges $K_{n,n}$ with two colours results in a monochromatic $K_{s,s}$-subgraph.

   (a) Prove that $R^{\text{bp}}(s) \leq 8^s$. (Note: this bound is not best-possible)

   (b) Prove that $R^{\text{bp}}(s) \geq 2^{s/2}$ for all $s \geq 2$.

6. Let G be a connected k-regular graph. Prove that G has exactly three distinct eigenvalues if and only if G is strongly regular. You may use the following two statements without proof:

   - For each connected k-regular graph H, k is an eigenvalue of H with multiplicity 1.
   - For every real symmetric $n \times n$ matrix M, there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of M.