GRAPH THEORY COMPREHENSIVE

16 June 2015, 1-4pm

Instructions: A complete examination consists of solutions to all six questions. All questions have equal value and, in multipart questions, all parts have equal value.

Problem 1. An *ear decomposition* of a graph G is a sequence $C, P_1, P_2, \ldots, P_k$ (possibly $k = 0$) in which C is a cycle of G and, for $i = 0, 1, 2, \ldots, k-1$, setting $H_i = C \cup (\bigcup_{j=1}^{i} P_i)$, P_{i+1} is a path in G having both ends in H_i but otherwise is disjoint from H_i.

1. Prove that every 2-connected graph has an ear decomposition.
2. Suppose $C, P_1, P_2, \ldots, P_k$ is an ear decomposition of a graph G such that C is an odd cycle and, for every $i = 1, 2, \ldots, k$, P_k is an odd length path. Prove that, for every vertex v of G, $G - v$ has a perfect matching.

Problem 2. A graph H is a *minor* of G if it is isomorphic to a graph that can be obtained from a subgraph of G by contracting edges. A graph H is a *topological minor* of G if G has a subgraph K such that an isomorph of H is obtained from K by contracting only edges that are incident with a vertex of degree 2.

Prove from first principles that, if G has at least one of $K_{3,3}$ or K_5 as a minor, then G has at least one of $K_{3,3}$ or K_5 as a topological minor.

Problem 3. State and prove from first principles Turán's Theorem about graphs not having K_n as a subgraph.

Problem 4. **(a)** Let G be a graph with no isolated vertices. Prove that if G is edge-transitive but not vertex-transitive, then G is bipartite.

(b) Prove that if a graph is both vertex- and edge-transitive, but not arc-transitive, then all its vertices have even degree.

Problem 5. State and prove from first principles Brooks' Theorem relating the chromatic number and maximum degree of a graph.

Problem 6. Prove from first principles Seymour's Theorem that every 2-edge-connected graph has a $(\mathbb{Z}_2 \times \mathbb{Z}_3)$-flow.