GRAPH THEORY COMPREHENSIVE

16 June 2015, 1-4pm

Instructions: A complete examination consists of solutions to all six questions. All questions have equal value and, in multipart questions, all parts have equal value.

Problem 1. An ear decomposition of a graph $G$ is a sequence $C, P_1, P_2, \ldots, P_k$ (possibly $k = 0$) in which $C$ is a cycle of $G$ and, for $i = 0, 1, 2, \ldots, k - 1$, setting $H_i = C \cup (\bigcup_{j=1}^{i} P_i)$, $P_{i+1}$ is a path in $G$ having both ends in $H_i$ but otherwise is disjoint from $H_i$.

1. Prove that every 2-connected graph has an ear decomposition.

2. Suppose $C, P_1, P_2, \ldots, P_k$ is an ear decomposition of a graph $G$ such that $C$ is an odd cycle and, for every $i = 1, 2, \ldots, k$, $P_k$ is an odd length path. Prove that, for every vertex $v$ of $G$, $G - v$ has a perfect matching.

Problem 2. A graph $H$ is a minor of $G$ if it is isomorphic to a graph that can be obtained from a subgraph of $G$ by contracting edges. A graph $H$ is a topological minor of $G$ if $G$ has a subgraph $K$ such that an isomorph of $H$ is obtained from $K$ by contracting only edges that are incident with a vertex of degree 2.

Prove from first principles that, if $G$ has at least one of $K_{3,3}$ or $K_5$ as a minor, then $G$ has at least one of $K_{3,3}$ or $K_5$ as a topological minor.

Problem 3. State and prove from first principles Turán’s Theorem about graphs not having $K_n$ as a subgraph.

Problem 4. (a) Let $G$ be a graph with no isolated vertices. Prove that if $G$ is edge-transitive but not vertex-transitive, then $G$ is bipartite.

(b) Prove that if a graph is both vertex- and edge-transitive, but not arc-transitive, then all its vertices have even degree.

Problem 5. State and prove from first principles Brooks’ Theorem relating the chromatic number and maximum degree of a graph.

Problem 6. Prove from first principles Seymour’s Theorem that every 2-edge-connected graph has a $(\mathbb{Z}_2 \times \mathbb{Z}_3)$-flow.