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The Gromov-Witten Potential of a Point, Hurwitz Numbers, and Hodge Integrals

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Abstract Hurwitz numbers, which count certain covers of the projective line (or, equivalently, factorizations of permutations into transpositions), have been extensively studied for over a century. The Gromov-Witten potential F of a point, the generating series for Hodge integrals on the moduli space of curves, has been a central object of study in Gromov-Witten theory. We define a slightly enriched Gromov-Witten potential G (including integrals involving one "λ-class"), and show that after a non-trivial change of variables, $G = H$ in positive genus, where H is a generating series for Hurwitz numbers. We prove a conjecture of Goulden and Jackson on higher genus Hurwitz numbers, which turns out to be an analogue of a genus expansion ansatz of Itzykson and Zuber. As consequences, we have new combinatorial constraints on F, and a much more direct proof of the ansatz of Itzykson and Zuber.

We can produce recursions and explicit formulas for Hurwitz numbers; the algorithm presented should prove "all" such recursions. Furthermore, there are many more recursions than previously suspected from geometry (and indeed they should exist in all genera); as examples we present surprisingly simple new recursions in genus up to 3 that are geometrically mysterious.

As we expect this paper also to be of interest to combinatorialists, we have tried to make it as self-contained as possible, including reviewing some results and definitions well known in algebraic and symplectic geometry, and mathematical physics.