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A Note on Lack of Strong Duality for Quadratic Problems with Orthogonal Constraints

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Abstract   The general quadratically constrained quadratic program (QQP) is an important modelling tool for many diverse problems. The QQP is a general NP hard, and numerically intractable. Lagrangian relaxations often provide good approximate solutions to these hard problems. Such relaxations are equivalent to semidefinite programming (SDP) relaxations and can be solved efficiently.

For several special cases of QQP, e.g. general convex quadratic programs and trust region subproblems (one quadratic constraint), the Lagrangian relaxation provides the exact optimal value. This means that there is a zero duality gap and the problem is tractable. Surprisingly, there is another class of nonconvex problems with zero duality gaps. It has recently been shown, [2, 1], that the special homogeneous QQP with objective function that arises in the trace formulation of the quadratic assignment problem, Trace $AXBX^T - 2CX^T$, and with quadratic constraints that correspond to the matrix orthogonality condition $XX^T = I$ (or the negative semidefinite condition $XX^T - I \leq 0$) can have nonzero duality gaps. However, this duality gap can be closed if one adds the seemingly redundant contraints $X^TX = I(X^TX - I \leq 0$, respectively), thus effectively doubling the number of constraints.

In this paper we show that the strong duality result does not hold for the more general inhomogeneous problem. In fact, strong duality will fail for the simple pure linear objective case. We also show how to close the duality gap for this pure linear case by relaxing the constraints to $XX^T + YY^T = I$, thus effectively doubling the number of variables.