

**CORR 2000-51**

**The Critical Group of a Directed Graph**

**David G. Wagner**

**Abstract** For a finite, directed graph  $G = (V, E)$  we define the critical group  $\mathcal{K}(G)$  to be the cokernel of the transpose of the Laplacian matrix of  $G$  acting on  $\mathbf{Z}^V$ , and  $K(G)$  to be its torsion subgroup. This generalizes the case of undirected graphs studied by Bacher, de la Harpe and Nagnibeda, and Biggs. We prove a variety of results about these critical groups, among which are: that  $\mathcal{K}(G/\pi)$  is a subgroup of  $\mathcal{K}(G)$  when  $\pi$  is an equitable partition and  $G$  is strongly connected; that  $K(G)$  depends only on the graphic matroid of  $G$  when  $G$  is undirected; that there is no ‘natural’ bijection between spanning trees of  $G$  and  $K(G)$  when  $G$  is undirected, even though these sets are equicardinal; and that the ‘dollar game’ of Biggs can be generalized slightly to provide a combinatorial interpretation for the elements of  $K(G)$  when  $G$  is strongly connected.