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Geometry of Semidefinite Max-Cut Relaxations via Ranks

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Abstract   Semidefinite programming (SDP) relaxations are proving to be a powerful tool for finding tight bounds for hard discrete optimization problems. This is especially true for one of the easier NP-hard problems, the Max-Cut problem (MC). The well-known SDP relaxation for Max-Cut, here denoted SDP1, can be derived by a first lifting into matrix space and has been shown to be excellent both in theory and in practice.

Recently the present authors have derived a new relaxation using a second lifting. This new relaxation, denoted SDP2, is strictly tighter than the relaxation obtained by adding all the triangle inequalities to the well-known relaxation. In this paper we present new results that further describe the remarkable tightness of this new relaxation. Let $\mathcal{F}_n$ denote the feasible set of SDP2 for the complete graph with n nodes, let F_n denote the appropriately defined projection of $\mathcal{F}_n$ into $mathcal S_n$, the space of real symmetric $n \times n$ matrices, and let C_n denote the cut polytope in $mathcal S_n$. Further let $Y \in \mathcal{F}_n$ be the matrix variable of the SDP2 relaxation and $X \in F_n$ be its projection. Then for the complete graph on 3 nodes, $F_3 = C_3$ holds. We prove that the rank of the matrix variable $Y \in \mathcal{F}_3$ of SDP2 completely characterizes the dimension of the face of the cut polytope in which the corresponding matrix X lies. This shows explicitly the connection between the rank of the variable Y of the second lifting and the possible locations of the projected matrix X within C_3. The results we prove for $n = 3$ cast further light on how SDP2 captures all the structure of C_3, and furthermore they are stepping stones for studying the general case $n \geq 4$. For this case, we show that the characterization of the vertices of the cut polytope via rank $Y = 1$ extends to all $n \geq 4$. More interestingly, we show that the characterization of the one-dimensional faces via rank $Y = 2$ also holds for $n \geq 4$. Furthermore, we prove that if rank $Y = 2$ for $n \geq 3$, then a simple algorithm exhibits the

two rank-one matrices (corresponding to cuts) which are the vertices of the one-dimensional face of the cut polytope where X lies.