Abstract

Double Hurwitz numbers count branched covers of $\mathbb{CP}^1$ with fixed branch points, with simple branching required over all but two points 0 and ∞, and the branching over 0 and ∞ points specified by partitions of the degree (with m and n parts respectively). Single Hurwitz numbers (or more usually, Hurwitz numbers) have a rich structure, explored by many authors in fields as diverse as algebraic geometry, symplectic geometry, combinatorics, representation theory, and mathematical physics. A remarkable formula of Ekedahl, Lando, M. Shapiro, and Vainshtein (the ELSV formula) relates single Hurwitz numbers to intersection theory on the moduli space of curves. This connection has led to many consequences, including Okounkov and Pandharipande's proof of Witten's conjecture (Kontsevich's theorem) connecting intersection theory on the moduli space of curves to integrable systems.

In this paper, we determine the structure of double Hurwitz numbers using techniques from geometry, algebra, and representation theory. Our motivation is geometric: we give strong evidence that double Hurwitz numbers are top intersections on a moduli space of curves with a line bundle (a universal Picard variety). In particular, we prove a piecewise-polynomiality result analogous to that implied by the ELSV formula. In the case $m = 1$ (complete branching over one point) and n is arbitrary, we conjecture an ELSV-type formula, and show it to be true in genus 0 and 1. The corresponding Witten-type correlation function has a better structure than that for single Hurwitz numbers, and we show that it satisfies many geometric properties, such as the string and dilaton equations, and a genus expansion ansatz analogous to that of Itzykson and Zuber. We give a symmetric function description of the double Hurwitz generating series, which leads to explicit formulae for double Hurwitz numbers with given m and n, as a function of genus. In the case where m is fixed but not necessarily 1, we prove a topological recursion on the corresponding generating series, which leads to closed-form expressions for double Hurwitz numbers and an analogue of the Goulden-Jackson polynomiality conjecture (an early conjectural variant of the ELSV formula).

Coupled with earlier work connecting double Hurwitz numbers to integrable systems, an ELSV-type formula would translate all of this structure on double Hurwitz numbers to the intersection theory of the universal Picard variety described earlier.