2018 Graph Theory Comprehensive

A complete paper consists of solutions to all six of the problems.

1. **Prove König's Theorem:** the size of a largest matching in a bipartite graph G is equal to the size of a smallest vertex cover of the edges.

2. **Prove Brooks' Theorem:** if G is a connected graph with maximum degree Δ, then either G has a proper colouring of its vertices with Δ colours, or G is either a complete graph or a cycle.

3. **Prove Turán's Theorem:** if G is a simple graph G with n vertices and r is a positive integer such that no subgraph of G is isomorphic to K_{r+1}, then $|E(G)| \leq |E(T_{n,r})|$ and equality holds if and only if $G = T_{n,r}$. (Here $T_{n,r}$ is the complete multipartite graph with n vertices, r parts, and each part has size either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$.)

4. Let G be a (multi)graph of maximum degree Δ. Let $e = v_0v_1$ be an edge of G. Let $k \geq \Delta + 1$ and suppose that G is not k-edge-colourable but that $G - e$ is k-edge-colourable. Let ϕ be a k-edge-colouring of $G - e$. For all $u \in V(G)$, let $\phi(u)$ denote the subset of $[k]$ that does not appear (in ϕ) on any edge incident with u.

   Suppose that $P = v_0v_1 \dots v_m$ is a path such that for all $i \geq 1$, there exists $j < i$ such that the colour $\phi(v_iv_{i+1})$ does not appear at v_j (i.e. is in $\phi(v_j)$).

   Prove: For every $i \neq j$, the set of colours not appearing at v_i is disjoint from the set of colours not appearing at v_j. That is, prove that, for every $i \neq j$, $\phi(v_i) \cap \phi(v_j) = \varnothing$.

   Hint: Use double induction, first on m, then on $m - j$ where v_j is missing a colour that is also missing at v_m (i.e. $\phi(v_m) \cap \phi(v_j) \neq \emptyset$). Use Kempe changes! It may be useful in some cases to also consider a second colour, in particular one missing at v_{j+1}.

5. Let H be a subgraph of a graph G. A walk W in G is *H-avoiding* if no edge of W and no internal vertex of W is in H. Define the relation $\sim$ on $E(G) \setminus E(H)$ by $e \sim e'$ if there is an H-avoiding walk in G containing both e and e'.

   Prove that $\sim$ is a transitive relation.

6. Let $\mathcal{F}$ be a collection of subtrees of a tree T, and let k be a positive integer.

   Prove at least one of the following holds:

   (a) there are k vertex disjoint trees in $\mathcal{F}$, or

   (b) there is a set X of $< k$ vertices in T that intersects each tree in $\mathcal{F}$.

   Hint: Consider the set of all edges e in T such that both components of $T - e$ contain a tree in $\mathcal{F}$; prove that this set forms a subtree of T; then consider a leaf of this subtree.