First-Stage PhD Comprehensive Examination

in

CONTINUOUS OPTIMIZATION

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MC 5479, Thursday, June 13, 2019, 1pm – 4pm **(3 hours)**

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1. Consider the univariate function

$$f(x) := \begin{cases} e^x & \text{for } x < 0, \\ x + 1 & \text{for } x \geq 0. \end{cases}$$

(a) Show that this function is convex. Any standard theorem that characterizes convexity of functions may be used.

(b) Show that the gradient of this function is Lipschitz continuous, and find L, the Lipschitz constant of the gradient.

(c) Since f' is Lipschitz continuous, Zoutendijk's theorem for minimizing f using steepest descent is applicable. State Zoutendijk's theorem and the conclusion as it applies to this function f.

(d) Suppose that $x_k = 0$ on the kth iteration of the steepest descent method. Identify an interval of positive width of choices for x_{k+1} that satisfy the Wolfe conditions. (Select any reasonable values for the constants appearing in Wolfe's conditions, and state what values you used.)

2. Let n be a positive integer.

   (a) Assume $f : \mathbb{R}^n \to (-\infty, +\infty]$ is positively 1-homogeneous. Show that it is subadditive if and only if it is convex.

   (b) Recall, if $f : \mathbb{R}^n \to \mathbb{R}$ satisfies

      (i) $f(x) > 0, \forall x \in \mathbb{R}^n \setminus \{0\}$,

      (ii) $f(u+v) \leq f(u) + f(v), \forall u, v \in \mathbb{R}^n$,

      (iii) $f(\lambda x) = |\lambda| f(x), \forall x \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}$,

   then f is a *norm on* $\mathbb{R}^n$. Let

$$\mathcal{F} := \{f : f \text{ is a norm on } \mathbb{R}^n\},$$

   and

$$\mathcal{C} := \{C \subset \mathbb{R}^n : C \text{ is compact, convex, } 0 \in \operatorname{int}(C) \text{ and } C = -C\}.$$

   Prove that (with γ being the gauge function)

$$f(\cdot) := \gamma(C; \cdot) \text{ and } C := \{x \in \mathbb{R}^n : f(x) \leq 1\}$$

   define a one-to-one correspondence between $\mathcal{F}$ and $\mathcal{C}$.

3. Let n be a positive integer and $f : \mathbb{R}^n \to (-\infty, +\infty]$ be convex.

   (a) Define the notion of *subgradient* of f at $\bar{x} \in \mathbb{R}^n$.

   (b) Define the notion of *subdifferential* of f at $\bar{x} \in \mathbb{R}^n$.

   (c) Let $f(x) := ||x||_\infty$. What is the subdifferential of f at a given $\bar{x} \in \mathbb{R}^n$? Prove your claims.

   For the remaining parts of this question, let m and n be positive integers such that $n \geq m + 1$, and consider linear programming problems in the form

$$(\mathrm{P}) \qquad \begin{array}{rl} p^* := \min & c^\top x \\ \text{s.t.} & Ax = 0 \\ & e^\top x = n \\ & x \in \mathbb{R}^n_+, \end{array}$$

   where A is a given full row rank m-by-n matrix, $c \in \mathbb{R}^n$ is also given. $e \in \mathbb{R}^n$ is the vector of all ones. Assume that $Ae = 0$, $p^* = 0$ and $c^\top e > 0$. In your answers, you may use the Fundamental Theorem of LP, provided you state it clearly and correctly. For every $q \in \mathbb{R}_{++}$, define

$$\phi_q(x) := \begin{cases} (q+1)\ln\left(c^\top x\right) - \ln\left(\min_j\{x_j\}\right), & \text{if } x \in \mathbb{R}^n_{++} \\ +\infty, & \text{otherwise,} \end{cases}$$

   and consider the optimization problem

$$(P_q) \qquad \begin{array}{rl} v(P_q) := \inf & \phi_q(x) \\ \text{s.t.} & Ax = 0 \\ & e^\top x = n. \end{array}$$

   (d) Prove that for every $q \in \mathbb{R}_{++}$, $v(P_q) = -\infty$ (i.e., (P_q) is unbounded). Further prove that for every $q \in \mathbb{R}_{++}$, (P) and (P_q) are equivalent (by stating a suitable definition of equivalence and proving it).
   Hint:

   - (P) has optimal solution(s),
   - (P_q) has feasible sequences in the domain of ϕ_q which certify unboundedness of (P_q).

   (e) Let $f : \mathbb{R}^n_{++} \to \mathbb{R}$ be defined by $f(x) := -\ln\left(\min_j\{x_j\}\right)$. What is the subdifferential of f at a given $\bar{x} \in \mathbb{R}^n_{++}$? Prove your claims.

4. Let $a_1, \ldots, a_n \in \mathbb{R}^d$ be given, and consider the following optimization problem in which $x_1, \ldots, x_n \in \mathbb{R}^d$ are the unknowns (i.e., nd total variables), $\lambda > 0$ is a fixed parameter, and all norms are Euclidean:

$$\min_{x_1,\ldots,x_n} f(x) := \frac{1}{2}\sum_{i=1}^{n} \|x_i - a_i\|^2 + \lambda \sum_{1 \le i < j \le n} \|x_i - x_j\|.$$

(Note: This formulation arises in "sum-of-norms" clustering.)

(a) What is the definition of "strongly convex"? Suppose $g, h : \mathbb{R}^n \to \mathbb{R}$ are two convex functions such that g is strongly convex. Prove that $g + h$ is also strongly convex.

(b) Apply the result in (a) to the function at hand to conclude that f is strongly convex. Justify your answer.

(c) By introducing auxiliary variables, rewrite the problem of minimizing f as a constrained optimization in standard conic form $\min c^T x$ subject to $Ax = b$, $x \in K$ (not necessarily the same x), where the convex cone K is a Cartesian product of second-order cones. The following standard trick may help: the convex quadratic constraint $s \ge \|x\|^2/2$ may be expressed in second-order cone form:

$$\frac{1}{2}(s+1) \ge \left\| \left[x; \frac{1}{\sqrt{2}}(s-1) \right] \right\|,$$

where the notation on the right-hand side indicates concatenation of a vector and scalar.