C&O Comprehensive
Graph Theory

Thursday 13 June, 1-4PM

Examiners: Penny Haxell and Peter Nelson

Instructions: A complete examination consists of solutions to all six questions. All questions have equal value and, in multipart questions, all parts have equal value. All proofs should be derived from first principles unless indicated otherwise. Where assumptions are allowed, **clearly state any assumptions you make**. All graphs are simple.

1. For each integer $s \geq 3$, let $R(s)$ denote the minimum integer n such that every colouring of the edges of K_n with two colours has a monochromatic K_s-subgraph.
   (a) Prove that $R(s) \leq 4^s$.
   (b) Prove that $R(s) \geq \sqrt{2}^s$.
2. State and prove Turán's theorem characterizing, for all integers $n \geq t \geq 2$, the graphs on n vertices with no K_t-subgraph having the maximum possible number of edges.
3. Prove that if G is a 3-connected graph embedded in the plane, then a cycle C of G is the boundary of a face if and only if C has no chord and the graph $G - V(C)$ is connected. You may assume standard results from graph connectivity and topology.
4. Let $k \geq 3$ be an integer. Prove that if G is a k-connected graph on at least three vertices having no $(k+1)$-vertex independent set, then G has a Hamilton cycle. You may assume standard results from graph connectivity.
5. (a) Prove that if G is an edge-transitive graph without isolated vertices that is not vertex-transitive, then G is bipartite.
   (b) Prove that if G is a vertex-transitive, edge-transitive graph that is not arc-transitive, then every vertex of G has even degree.
6. Prove that every 2-edge-connected graph has a nowhere-zero $(\mathbb{Z}_2 \times \mathbb{Z}_3)$-flow.