Some Applications of Symmetric Cone Programming in Financial Mathematics

by

AO LI

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Abstract

Proper forecasting of uncertain parameters of the problems is one important application of *convex optimization*. Currently, there are many successful approaches in the theory and algorithms to solve convex programming problems.

One of the most popular applications is in *Financial Markets*. We present some fundamental SDP representation techniques for moment problems and introduce portfolio optimization problems. We also revise one portfolio optimization model and use numerical results to show the advantage of revised model.

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Chapter 1

Introduction

One of the most important ingredients of successful applications of optimization is the proper forecasting of uncertain parameters of the problem. In many cases, we deal with uncertainties in data and parameters by estimating high order expected behavior. Indeed, in most cases when there is significant uncertainty, the expected value of a parameter is a poor way to represent the real problem as a mathematical, deterministic optimization problem.

Research in optimization under uncertainty has been flourishing during the last two decades. The main approaches are covered under the terms: *Stochastic Programming* and *Robust Optimization*.

Applications in the area of optimization under uncertainty have also been increasing in number as well as in practical impact. One of the most popular and visible applications is in *Financial Markets*.

This essay is geared towards financial applications. In such applications, many restrictions on the variables based on variance data can be expressed as variable vectors lying in well-behaved convex cones. Many other restrictions based on higher order moments can be expressed as certain scalar polynomials being nonnegative for every choice of its argument. Such positivity requirements can be equivalently expressed as certain variable matrix being symmetric positive definite.

In this essay, we first review these fundamental representation techniques (see Section 1.1 and Chapter 2). All the convex cones used in our formulations are unified under a well-behaved set of convex cones called *symmetric cones* (see the next section for a definition). We then turn to the financial applications and introduce portfolio optimization (see Section 1.2 and 2.2).

In Chapter 3, we focus on the portfolio optimization model proposed by Lobo et al. [7]. We modify and improve their model in two ways:

1. We allow for cash infusions in each planning period.

2. We allow for multiple periods.

In Chapter 4, we compare the performance of Lobo et al. model and our modifications using real data and computational experiments.

1.1 Symmetric Cone Optimization

Convex optimization problems, the problems of minimizing a convex function over a convex set, make up a very large and relatively well-behaved class of optimization problems.

Currently, many of the most successful approaches in the theory and algorithms treat convex optimization problems in *conic form.* A popular name for such form is *cone programming problems.*

Cone programming problem is the problem of optimizing (minimizing or maximizing) a linear function of finitely many real variables subject to the vector of real variables lying in the intersection of a prescribed affine subspace and a convex cone. Below, we introduce some notation and describe the cone programming problems in terms of the notation.

Given $\mathcal{A} : \mathbb{R}^n \mapsto \mathbb{R}^m$ a linear operator that is surjective, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, consider the cone programming problem

$$\begin{array}{llrcl} (P) & \inf & \langle c, x \rangle & & \\ & & \mathcal{A}(x) & = & b, \\ & & x & \in & K, \end{array}$$

where $K \subset \mathbb{R}^n$ is a closed convex cone.

Note that $K \subset \mathbb{R}^n$ is a *cone* if $\forall x \in K$ and $\forall \lambda > 0$, $\lambda x \in K$.

Under very mild assumptions, all convex optimization problems can be formulated as cone programming problems, see for instance [16].

Any linear operator $\mathcal{A} : \mathbb{R}^n \mapsto \mathbb{R}^m$ can be represented by m elements of $\mathbb{R}^n$. That is, there exist $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m \in \mathbb{R}^n$ such that:

$$[\mathcal{A}(x)]_i = \langle \mathcal{A}_i, x \rangle, \ \forall i \in \{1, 2, \ldots, m\}.$$

Then $\mathcal{A}$ being surjective is equivalent to $\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m\}$ being linearly independent. Indeed the latter condition can be easily checked.

We denote by $\mathbb{S}^n$ the space of $n \times n$ symmetric matrices over the reals. $\mathbb{S}^n_+$ denotes the cone of symmetric, positive semidefinite matrices in $\mathbb{S}^n$. In the above optimization problem, setting

$$K := \mathbb{S}^{n_1}_+ \oplus \mathbb{S}^{n_2}_+ \oplus \cdots \oplus \mathbb{S}^{n_r}_+$$

yields a *Semidefinite Programming*(SDP) problem, where $\oplus$ denotes the *direct sum* of two vector spaces V and W. The direct sum $V \oplus W$ is the set of vectors (v, w), $v \in V$ and $w \in W$, with the operations

$$(v,\, w) + (v',\, w') = (v + v', w + w'),\; c(v, w) = (cv, cw).$$

Many financial applications can be treated via second order cones. An $(n+1)$-dimensional *second order cone* is defined as

$$SOC^n := \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R} \oplus \mathbb{R}^n :\; x_0 \geq \|x\|_2 \right\}.$$

We also call the cone $A(SOC^n)$ a second order cone, for every nonsingular linear transformation $A : \mathbb{R}^{n+1} \mapsto \mathbb{R}^{n+1}$. For example the cone

$$cl \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R} \oplus \mathbb{R}^n :\; x_0 x_1 > x_2^2 + x_3^2 + \cdots + x_n^2,\; x_0 > 0 \right\}$$

is equal to the image of SOC^n under such nonsingular linear transformation.

If we use

$$K := SOC^{n_1} \oplus SOC^{n_2} \oplus \cdots \oplus SOC^{n_k},$$

then we have a *Second Order Cone Programming* (SOCP) problem.

Note that every cross section of SOC^n with $x_0 := \alpha > 0$ gives an Euclidean Ball in $\mathbb{R}^n$. This cone is also called the *ice-cream cone*, *light cone* or *Lorentz cone.*

The cone $K \subseteq \mathbb{R}^n$ is defined to be *pointed* if $K \bigcap (-K) = \{0\}$, which is equivalent to say that K contains no lines.

Given $K \subseteq \mathbb{R}^n$, the *dual* cone of K is

$$K^* \;:= \{s \in \mathbb{R} :\; \langle x, s\rangle \geq 0, \forall x \in K\}.$$

If cone $K \subseteq \mathbb{R}^n$ has nonempty interior:

- K is *homogeneous* if for every pair $x, \ y \in int(K)$ (denoting the interior of K), there exists a nonsingular and linear transformation T such that , $T(K) = K$ and $T(y) = x$.
- K is *self-dual* if there exists an inner product under which, $K = K^*$.
- K is *symmetric* if it is homogeneous and self-dual.

Now, we list some fundamental results on some of the elementary properties of convex cones.

Theorem 1.1.1. *Let $K \subseteq \mathbb{R}^n$. If K is a pointed, closed convex cone with nonempty interior, then so is K^*.*

Theorem 1.1.2. *Let $K \subseteq \mathbb{R}^n$. Then K is a closed convex cone iff $K^{**} = K$.*

Corollary 1.1.1. *Let $K \subseteq \mathbb{R}^n$. Then K is a pointed, closed convex cone with nonempty interior iff so is K^*.*

Theorem 1.1.3. *Let $K \subseteq \mathbb{R}^n$ be a closed convex cone. Then*

$$int(K) \ = \{x \in K : \ \langle x, s \rangle > 0, \ \forall s \in K^* \backslash \{0\}\}.$$

Both $\mathbb{S}^n_+$ and the SOC^n are pointed, closed, convex cones with nonempty interior. Moreover, they are homogeneous and self-dual; hence, they are symmetric.

For the rest of the essay, the main convex cones we deal with will be $\mathbb{S}^n_+$ and SOC^n. Most of the models will only use second order cones. However, if we want to include more complicated constraints in our model, such as simple polynomial inequalities stipulating that a scalar polynomial be nonnegative, then we would utilize the results of Chapter 2 and the cone of symmetric positive definite matrices.

1.2 Portfolio Optimization

In his seminal paper [11], Markowitz introduced mathematical modelling techniques to solve the portfolio selection problem for a large private investor or an institutional investor. Markowitz's work provided a starting point for most of the work in the area of portfolio optimization.

In the book [10], Markowitz analyzed the Mean-Variance (M-V) portfolio selection problem. The problem is modelled as a parametric quadratic programming problem with general linear inequality constraints:

$$\begin{aligned} &\textit{minimize} && f(x) := -t\mu^T x + \frac{1}{2}x^T V x \\ &\textit{subject to} && Ax \le b \\ & && c_1 \le x \le c_2, \end{aligned}$$

where μ is an n-vector of expected returns, V is an $n \times n$ covariance matrix, x is an n-vector of amount of asset holdings, A is an $m \times n$ matrix, b is an m-vector, t is a parameter (usually $t \ge 0$) and $c_1, c_2 \in \mathbb{R}^n$ are bounds on the holdings. Note that the objective function is quadratic and the constraints are linear. The function $f(x)$ is called the utility function in [10].

An obvious drawback of the basic Markowitz model is that it needs the mean μ and variance V computed (estimated), and then uses μ, V in a deterministic quadratic programming setting. During the last decade, a new area called *robust optimization* provided a very intriguing approach to uncertainty in optimization problems [1].

Instead of using a single estimate of an uncertain part of the data (or parameters) in the Mean-Variance(M-V) portfolio selection model, robust optimization approach describes a set of possible values for that uncertain data (or parameters), which is called the *uncertainty region*. Then the robust portfolio selection problems try to find the optimal strategy under the assumption that the worst possible scenario in the uncertainty region can happen [9].

Many financial investment companies use the notion of *market driving factors* in their forecasting techniques. They choose a small set of indicators (order of 10 or at most 20) that reflect the basic tendencies of the financial market. Below, vector f in the *Factor analysis model* is the vector representing such market driving factors.

Define the return

$$r := \mu + V^T f + \varepsilon,$$

where $\mu \in \mathbb{R}^n$ is the vector of expected returns, $f \sim \aleph(0, F) \in \mathbb{R}^m$ is the vector of returns of the factors that drive the market, $V \in \mathbb{R}^{m \times n}$ is the matrix of factor loadings of the n assets and $\varepsilon \sim \aleph(0, D)$ is the vector of residual returns, where $x \sim \aleph(a, A)$ denotes that x is a multivariate normal random variable with mean vector a and covariance matrix A. Then

$$r \sim \aleph(\mu, V^T F V + D).$$

Goldfarb and Iyengar [9] noted that the eigenvalues of the residual covariance matrix D are typically much smaller than those of the covariance matrix V^TFV implied by the factors. Thus, the covariance matrix of return r is usually dominated by V^TFV. In some cases, the lower-rank property of F and V can reduce the complexity of calculation for the covariance matrix.

Let the portfolio of the investor be represented by $\phi \in \mathbb{R}^n$, where ϕ_i is the amount invested for asset i. The return r_ϕ for the portfolio can be given by

$$r_\phi = r^T\phi = \mu^T\phi + V^Tf\phi + \varepsilon^T\phi \sim \aleph(\mu^T\phi, \phi^T(V^TFV + D)\phi).$$

Define the uncertainty regions as:

$$\begin{aligned} S_d &:= \{D: \ D = Diag(d), d_i \in [\underline{d}_i, \bar{d}_i], i = 1, \ldots, n\} \\ S_v &:= \{V: \ V = V_0 + W, \|W_i\|_g \leq \rho_i, i = 1, \ldots, n\} \\ S_m &:= \{\mu: \ \mu = \mu_0 + \xi, |\xi_i| \leq \gamma_i, i = 1, \ldots, n\}, \end{aligned}$$

where W_i is the ith column of W and $\|w\|_g = \sqrt{w^TGw}$ is the one kind of norm for w with respect to a symmetric, positive definite matrix G.

Mathematically, Goldfarb and Iyengar in [9] expressed the robust portfolio selection model that minimizes the variance $Var[r_\phi]$ among all $\phi \in \mathbb{R}^n$ that have expected return $E[r_\phi]$ at least α as:

$$\begin{array}{lcl} minimize & max_{\{V\in S_v, D\in S_d\}} & Var[r_\phi] \\ subject\ to & min_{\{\mu\in S_m\}} & E[r_\phi] \geq \alpha \\ & & e^T\phi = 1. \end{array}$$

Another closely related model for the investor to maximize the expected return is:

$$\begin{array}{lcl} maximize & min_{\{\mu\in S_m\}} & E[r_\phi] \\ subject\ to & max_{\{V\in S_v, D\in S_d\}} & Var[r_\phi] \leq \lambda \\ & & e^T\phi = 1. \end{array}$$

Assume that the uncertainty sets S_v and S_m are finite sets[9], i.e.

$$S_v = \{V_1, \ldots, V_s\} \ \ S_m = \{\mu_1, \ldots, \mu_r\}.$$

By series of modelling theorems and techniques, the model can be reformulated as an instance of SOCP.

The portfolio selection model we talk in Chapter 3 is arising from the portfolio optimization problems with fixed transactions costs, which are cast as convex optimization problems. While convex portfolio optimization problems include those with linear transactions costs, margin and diversification constraints, and limits on variance and on shortfall risk.

The model can be view as an extension of the basic Markowitz model. Instead of defining objective function with variance, we treat the variance as a second order cone constraint. We also express shortfall risk, which is equal to VaR in real applications, in constraints.

We do not employ the robust portfolio selection approach in this essay. However, as we point out in Chapter 3, our formulations can be extended to the robust optimization model by introducing uncertainty regions.

The recent theoretical and computational developments in SDP and SOCP, especially interior-point methods, provide us with fast algorithms, good modelling techniques and robust software for many nonlinear convex optimization problems. Our model is numerically solvable under these advances.

Chapter 2

SDP Representation of Positive Polynomials for Moment problems

Moment problems, involving the first k order moment of random variables, have proven to be applicable to different areas such as computational finance, operations research in general, and stochastic optimization. Employing duality theory and other representation tools, SDP can be used to represent the moment-type optimization problems. In this chapter, we will show the key connections between moment problems, positive polynomials and SDP representations. In Section 2.1, some fundamental theorems and results are outlined. In Section 2.2, some examples of the financial applications are introduced.

2.1 Mathematical Foundations of SDP Representation

Nesterov [12] showed that the set of coefficients of a degree n univariate polynomial, which generate polynomials with non-negative values for every choice of the argument can be represented as an intersection of the positive semidefinite cone with an affine space. Here, we outline this result and some related theory.

$\mathcal{P}^n$ denotes the $(n+1)-$dimensional vector space. $p \in \mathcal{P}^n$ is written as

$$p(t) \;=\; \sum_{k=0}^{n} p_k t^k.$$

Defining

$$\tau_n \ := \ (1, t, t^2, \dots, t^n)^T \in \mathcal{P}^n,$$

we have $p(t) = \langle p, \tau_n \rangle$.

The cone of non-negative polynomials is the cone of all coefficient vectors p for which the underlying polynomial is non-negative for all values of t. It is easy to show that such a polynomial must be of even degree. That is

$$K_{2n} \ := \ \left\{ p \in \mathcal{P}^{2n} : \ p(t) \geq 0, \text{ for all } t \in \mathbb{R} \right\}.$$

Theorem 2.1.1. *[12]*

(i) A polynomial of odd degree cannot be nonnegative on $\mathbb{R}$. That is, if $p(t) \geq 0$, $\forall t \in \mathbb{R}$, then the degree of $p(t)$ is even.

(ii) A polynomial is nonnegative on the whole of $\mathbb{R}$ iff the polynomial can be expressed as a sum of squares of polynomials.

Proof. We use $\iota := \sqrt{-1}$

(i) We give two proofs for this part:

(a) Suppose $p(t)$ has odd degree, $deg(p) = 2k+1$ for some $k \in \mathbb{Z}_+$.
The coefficient of t^{2k+1} is nonzero by definition, and denote it by p_{2k+1}.
If $p_{2k+1} > 0$, then $p_{2k+1}t^{2k+1} < 0$ for all sufficient small $t (t \to -\infty)$.
If $p_{2k+1} < 0$, then $p_{2k+1}t^{2k+1} < 0$ for all sufficient large $t (t \to +\infty)$.
Therefore, $p(t)$ is not nonnegative on the whole $\mathbb{R}$ when $p_{2k+1}t^{2k+1}$ dominates $p(t)$.

(b) Assume $p(t) \geq 0$ for all $t \in \mathbb{R}$. Let λ_i be its real roots with multiplicity m_i, for $i \in \{1, \dots, r\}$, and $a_j + \iota b_j, a_j - \iota b_j$ be its complex roots for $j \in \{1, \dots, h\}$.
Then

$$p(t) = p_n \prod_{i=1}^{r} (t - \lambda_i)^{m_i} \prod_{j=1}^{h} ((t - a_j)^2 + b_j^2).$$

If m_i is odd, the sign of $(t - \lambda_i)^{m_i}$ is different for $t \in (\lambda_i, \lambda_i + \xi)$ and $t \in (\lambda_i - \xi, \lambda_i)$, where $\xi > 0$ is sufficiency small. The sign of $p(t)$ is also different when the sign of the other items remains unchanged. This is a contradiction to $p(t) \geq 0$ for all $t \in \mathbb{R}$. So m_i must be even for $i \in \{1, \dots, r\}$.
The degree of $p(t)$ is $(m_1 + m_2 + \ldots + m_r) + 2h$ must also be even.

(ii) If $p(t)$ can be expressed as a sum of squares of polynomials, then clearly $p(t) \geq 0$ for all $t \in \mathbb{R}$.

If $p(t) \geq 0$ for all $t \in \mathbb{R}$, we use the same notation of roots of $p(t)$ as above, and let $2k$ denote the degree of p. Then

$$\begin{aligned}
p(t) &= p_{2k} \prod_{i=1}^{r} (t-\lambda_i)^{m_i} \prod_{j=1}^{h} ((t-a_j)^2 + b_j^2) \\
&= p_{2k}[(t-\lambda_1)^{m_1/2}(t-\lambda_2)^{m_2/2} \cdots (t-a_1) \cdots (t-a_h)]^2 \\
&\quad + p_{2k}[(t-\lambda_1)^{m_1/2} \cdots (t-a_{h-1})b_h]^2 \\
&\quad + p_{2k}[(t-\lambda_1)^{m_1/2} \cdots (t-a_{h-2})b_{h-1}b_h]^2 \\
&\quad + \cdots \\
&= \sum_{i=0}^{k} (\sum_{j=0}^{k} c_{ij} t^j)^2.
\end{aligned}$$

Where c_{ij} is the coefficient for x^j in the ith polynomial of the sum. Then $p(t)$ can be expressed as a sum of squares of polynomials.

■

As it will become clearer, it seems more natural to treat such cones as squares of some other object. There are many versions of this. For instance, we may be interested only in non-negative values of t or only in $t \in [0, 1]$, etc.

Let E_k denote the kth $(n+1) \times (n+1)$ cross diagonal matrix:

$$E_0 := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad E_1 := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$E_2 := \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \dots, E_{2n} := \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Theorem 2.1.2. *[12]*

$$K_{2n} = \{p \in \mathcal{P}^{2n} : \; p_k = \langle X, E_k \rangle, \; k \in \{0, 1, \ldots, 2n\}, \;\; X \succeq 0\}.$$

Proof.

Let $T := \{p \in \mathcal{P}^{2n} : \; p_k = \langle X, E_k \rangle, \; k \in \{0, 1, \ldots, 2n\}, \;\; X \succeq 0\}$, we will prove that $T = K_{2n}$.

(i) For all $p \in T$, $X := (x_{ij})$, $p_k = \langle X, E_k \rangle = \sum_{i+j=k+2} x_{ij}$,

$$\begin{aligned} p(t) &= \sum_{k=0}^{2n} p_k t^k = \sum_{k=0}^{2n} \sum_{i+j=k+2} x_{ij} t^k \\ &= \tau_{2n}^T X \tau_{2n}. \end{aligned}$$

As $X \succeq 0$, we have $\tau_{2n}^T X \tau_{2n} \geq 0$ for $\forall t \in \mathbb{R}$.

Hence, $p(t) \geq 0$ for $\forall t \in \mathbb{R}$. Therefore, $T \subseteq K_{2n}$.

(ii) Using the same notation as in the proof of the Theorem 2.1.1, for $\forall p \in K_{2n}$, $p(t) = \sum_{i=0}^{n} (\sum_{j=0}^{n} c_{ij} t^j)^2$.

Let C be the $(n+1) \times (n+1)$ matrix whose (ij)th entry is c_{ij}.

Define $X := C^T C$, note that $\sum_{j=0}^{n} c_{ij} t^j = (C\tau_n)_i$, we can obtain

$$\begin{aligned} p(t) &= \tau_n^T C^T C \tau_n = \tau_n^T X \tau_n \\ &= \sum_{k=0}^{2n} (\sum_{i+j=k+2} x_{ij}) t^k. \end{aligned}$$

Thus, $p_k = \sum_{i+j=k+2} x_{ij} = \langle X, E_k \rangle$, and $X = C^T C \succeq 0$.

So $p \in T$. Therefore, $K_{2n} \subseteq T$.

Combining (i) and (ii), we conclude that $K_{2n} = T$.

■

Lemma 2.1.1. $(K_{2n})^* = \{s \in \mathcal{P}^{2n} : \sum_{k=0}^{2n} s_k E_k \succeq 0\}$.

Proof.

Let $M := \{s \in \mathcal{P}^{2n} : \sum_{k=0}^{2n} s_k E_k \succeq 0\}$.

By definition, $(K_{2n})^* = \{s \in \mathcal{P}^{2n} :\ \langle p, s\rangle \geq 0, \forall p \in K_{2n}\}$.

(i) For $\forall s \in M, \forall p \in K_{2n}$, $\sum_{k=0}^{2n} s_k E_k \succeq 0$, $\langle p, s\rangle = \sum_{k=0}^{2n} p_k s_k$.
By Theorem 2.1.2, $p_k = \langle X, E_k\rangle, X \succeq 0$. Thus,

$$\langle p, s\rangle = \sum_{k=0}^{2n}\langle X, E_k\rangle s_k = \sum_{k=0}^{2n}\langle X, s_k E_k\rangle = \langle X, \sum_{k=0}^{2n} s_k E_k\rangle \geq 0,$$

since $X \succeq 0$, and $\sum_{k=0}^{2n} s_k E_k \succeq 0$. Therefore $s \in (K_{2n})^*$ and $M \subseteq (K_{2n})^*$.

(ii) For $\forall s \in (K_{2n})^*, \forall p \in K_{2n}$, $\langle p, s\rangle \geq 0$. Let $\forall a \in \mathbb{R}^{n+1}$, define $X\ := aa^T \succeq 0$. Let $p_k = \langle E_k, X\rangle$. Therefore, $p \in K_{2n}$ by Theorem 2.1.2.

$$\begin{aligned} a^T(\sum_{k=0}^{2n} s_k E_k)a &= \sum_{k=0}^{2n} s_k a^T E_k a = \sum_{k=0}^{2n} s_k trace(a^T E_k a) \\ &= \sum_{k=0}^{2n} s_k \langle E_k, aa^T\rangle = \sum_{k=0}^{2n} s_k p_k \\ &= \langle p, s\rangle \geq 0. \end{aligned}$$

Note that the equation above is true for $\forall a \in \mathbb{R}^{n+1}$, thus $\sum_{k=0}^{2n} s_k E_k \succeq 0$, implying $s \in M$. Hence $(K_{2n})^* \subseteq M$.

Therefore, $(K_{2n})^* = M = \{s \in \mathcal{P}^{2n} : \sum_{k=0}^{2n} s_k E_k \succeq 0\}$.

■

Lemma 2.1.2. $int(K_{2n}) = \{p \in \mathcal{P}^{2n} : p_{2n} > 0, p(t) > 0, \forall t \in \mathbb{R}\}$.

Proof.

By Theorem 1.1.3, $int(K_{2n})\ = \{p \in K_{2n},\ \langle p, s\rangle > 0,\ \forall s \in K_{2n}^* \backslash \{0\}\}$.

Let $p \in int(K_{2n})$. For $\bar{s} := (0, 0, \ldots, 0, 1)^T \in \mathcal{P}^{2n}$,

$$\sum_{k=0}^{2n} s_k E_k = E_{2n} \succeq 0 \implies \bar{s} \in K_{2n}^* \backslash \{0\}$$
$$\implies \langle p, \bar{s} \rangle = p_{2n} > 0.$$

Select $\hat{s} := (1, t, t^2, \ldots, t^{2n})^T \in \mathcal{P}^{2n}$,

$$\tilde{p}^T (\sum_{k=0}^{2n} s_k E_k) \tilde{p} = \sum_{k=0}^{2n} s_k \tilde{p}^T E_k \tilde{p} = \tilde{p}(t)^2 \geq 0,$$

is true for $\forall \tilde{p} \in \mathcal{P}^{2n}$.

Then $\sum_{k=0}^{2n} \hat{s}_k E_k \succeq 0$, $\hat{s} \in K_{2n}^* \backslash \{0\}$, therefore $p(t) = \langle p, \hat{s} \rangle > 0$, for $\forall t \in \mathbb{R}$. We proved,

$$int(K_{2n}) \subseteq \{p \in \mathcal{P}^{2n} : p_{2n} > 0, p(t) > 0, \forall t \in \mathbb{R}\}.$$

In order to prove the converse, let $p \in \mathcal{P}^{2n}$ such that $p_{2n} > 0, p(t) > 0, \forall t \in \mathbb{R}$. We will prove that $\exists \bar{\varepsilon} > 0$, for every $h \in \mathcal{P}^{2n}$ such that $\parallel h \parallel = 1$ and $\forall t \in \mathbb{R}$,

$$(p \pm \varepsilon h)(t) > 0, \; \forall \varepsilon \in [0, \bar{\varepsilon}].$$

It suffices to prove that for every $i \in \{0, 1, 2, \ldots, 2n\}$ and $\forall t \in \mathbb{R}$,

$$(p \pm \varepsilon e_i)(t) > 0,$$

since $1, t, t^2, \ldots, t^{2n}$ make up a basis for the polynomials of degree at most $2n$. Since $p(t) > 0$, $\forall t \in \mathbb{R}$, there exists $\underline{\varepsilon} > 0$ such that $p(t) \geq \underline{\varepsilon}, \forall t \in \mathbb{R}$. Moreover, this implies that $\forall \varepsilon \in [0, \underline{\varepsilon})$ and $\forall t \in \mathbb{R}$,

$$(p \pm \varepsilon e_0)(t) = p(t) \pm \varepsilon > 0.$$

Define $\tilde{p} \in \mathcal{P}^{2n}$, such that $\tilde{p}(t) = p_{2n} + p_{2n-1} t + \ldots + p_0 t^{2n}$ for $t \in \mathbb{R}$, then

$$p(t) \;\; = \;\; t^{2n} \tilde{p}(\frac{1}{t}) = t^{2n}(p_{2n} + p_{2n-1} t^{-1} + \ldots + p_0 t^{-2n}). \qquad (2.1)$$

Since $p(t) > 0, \forall t \in \mathbb{R}$, we have $\tilde{p}(\frac{1}{t}) > 0$ for $\forall t \in \mathbb{R} \backslash \{0\}$. Thus $\tilde{p}(t) > 0, \forall t \in \mathbb{R}$. Therefore, there exists $\varepsilon_0 > 0$ such that $\tilde{p}(t) \geq \varepsilon_0, \forall t \in \mathbb{R}$. Using (2.1) we conclude that:

$$p(t) \geq \varepsilon t^{2n}, \; \forall \varepsilon \in [0, \varepsilon_0], \forall t \in \mathbb{R}.$$

Therefore, for $\forall t \in \mathbb{R}$, $\forall \varepsilon \in [0, \varepsilon_0)$,

$$(p \pm \varepsilon e_{2n})(t) \;=\; p(t) \pm \varepsilon t^{2n} > 0.$$

Let $\bar{\varepsilon} := min\{\varepsilon_0, \underline{\varepsilon}\}$, then $\forall \varepsilon \in [0, \bar{\varepsilon})$, $\forall i \in \{1, 2, \ldots, 2n-1\}$, for large t, $|\varepsilon t^i| < \varepsilon t^{2n}$, then,

$$(p \pm \varepsilon e_i)(t) > p(t) - \varepsilon t^{2n} > 0.$$

If $|t| < 1$, $(p \pm \varepsilon e_i)(t) > p(t) - \varepsilon \geq \underline{\varepsilon} - \varepsilon > 0$.

Hence, the converse is satisfied. Then $int(K_{2n}) = \{p \in \mathcal{P}^{2n} : p_{2n} > 0, p(t) > 0, \forall t \in \mathbb{R}\}$.

■

Theorem 2.1.3. *K_{2n} and K_{2n}^* are pointed, closed, convex cones with nonempty interiors.*

Proof.

Note that by Lemma 2.1.2, $int(K_{2n})$ is not empty. Since

$$K_{2n} = \{p \in \mathcal{P}^{2n} : \; p(t) = \langle p, \tau_{2n} \rangle \geq 0, \forall \tau_{2n} \in \mathcal{P}^{2n}\}$$

which is the intersection of infinitely many closed half spaces, K_{2n} is closed and convex.

K_{2n} is also pointed, otherwise we can find a line such that

$$\begin{aligned} \langle p + \alpha d, \tau_{2n} \rangle \geq 0, \; \forall \tau_{2n} \in \mathcal{P}^{2n} \;&\Longrightarrow\; \langle d, \tau_{2n} \rangle = 0, \; \forall \tau_{2n} \in \mathcal{P}^{2n} \\ &\Longrightarrow\; t = 0, \end{aligned}$$

which is a contradiction to $\forall t \in \mathbb{R}$.

Applying Corollary 1.1.1, we obtain that K_{2n}^* is also a pointed, closed, convex cone with nonempty interior.

■

Here, we give a direct argument proving that K_{2n}^* has nonempty interior.

Lemma 2.1.3. *There exists $\bar{s} \in \mathcal{P}^{2n}$ such that $\sum_{k=0}^{2n} \bar{s}_k E_k \succ 0$.*

Proof. Let $t_0 < t_1 < \ldots < t_n \in \mathbb{R}$. Define

$$\bar{s} \ := \begin{pmatrix} 1 & 1 & \ldots & 1 \\ t_0 & t_1 & \ldots & t_n \\ t_0^2 & t_1^2 & \ldots & t_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_0^{2n} & t_1^{2n} & \ldots & t_n^{2n} \end{pmatrix} e \in \mathcal{P}^{2n}.$$

Take $\forall p \in \mathcal{P}^{2n} \neq 0$,

$$p^T(\sum_{k=0}^{2n} s_k E_k)p = \sum_{k=0}^{2n} s_k p^T E_k p = \sum_{i=0}^{n} p(t_i)^2 > 0,$$

because such a polynomial p can have at most n real roots unless $p \equiv 0$. Hence, $\bar{s} \in int(K_{2n}^*)$, K_{2n}^* has nonempty interior and $\sum_{k=0}^{2n} s_k E_k \succ 0$.

■

Theorem 2.1.4. *If $p \in\ int(K_{2n})$ then the set*

$$\{X \succeq 0 :\ \langle X, E_k \rangle = p_k, \quad \textit{for all } k \in \{0, 1, \ldots, 2n\}\}$$

is bounded and there exists $X \succ 0$ such that $\langle X, E_k \rangle = p_k$ for all $k \in \{0, 1, \ldots, 2n\}$.

Proof.

Fix $p \in\ int(K_{2n})$, and let $\bar{X} \in \{X \succeq 0 :\ \langle X, E_k \rangle = p_k, \quad \text{for all } k \in \{0, 1, \ldots, 2n\}\}$. Select $\bar{s} \in int(K_{2n}^*)$, and define $\bar{S} := \sum_{k=0}^{2n} \bar{s}_k E_k > 0$,

$$\langle \bar{X}, \bar{S} \rangle = \langle \bar{X}, \sum_{k=0}^{2n} \bar{s}_k E_k \rangle = \sum_{k=0}^{2n} \bar{s}_k p_k.$$

Since p and $\bar{s}$ are fixed, $\sum_{k=0}^{2n} \bar{s}_k p_k = \ constant > 0$. Then for every $\bar{X}$ as above, it satisfies $\langle \bar{X}, \bar{S} \rangle = constant$. Note that

$$\left\{ X \succeq 0 :\ \langle X, \bar{S} \rangle = \sum_{k=0}^{2n} \bar{s}_k p_k = \ constant \right\}$$

is compact. Therefore, $\{X \succeq 0 :\ \langle X, E_k \rangle = p_k, \quad \text{for all } k \in \{0, 1, \ldots, 2n\}\}$ is bounded for every $p \in int(K_{2n})$.

By Lemma 2.1.1, $(K_{2n})^* = \{s \in \mathcal{P}^{2n} : \sum_{k=0}^{2n} s_k E_k \in \mathbb{S}_+^n\}$. By Lemma 2.1.3, $\exists \bar{s} \in K_{2n}^*$ such that $\sum_{k=0}^{2n} s_k E_k \in \mathbb{S}_{++}^n$. Besides, by Theorem 2.1.3, K_{2n} and K_{2n}^* are pointed, closed, convex cones with nonempty interiors. Therefore by Theorem 2.1.2,

$$K_{2n} = K_{2n}^{**} = \{\mathbb{A}(X) : X \in \mathbb{S}_+^n\},$$

where $\mathbb{A} : \mathbb{S}^n \longrightarrow \mathbb{R}^{2n+1}$ such that $[\mathbb{A}(X)]_i = \langle E_i, X\rangle$, $\forall i \in \{0, 1, \ldots, 2n\}$. Then,

$$K_{2n}^* = \{s : \mathbb{A}^*(s) \in \mathbb{S}_+^n\}.$$

($\mathbb{A}^*$ denotes the adjoint of $\mathbb{A}$, such that

$$\langle \mathbb{A}^*(s), X\rangle = \langle s, \mathbb{A}(X)\rangle, \forall X \in \mathbb{S}^n, s \in \mathbb{R}^{2n+1}).$$

By standard duality theory, there exists $\bar{X} \in \mathbb{S}_{++}^n$ such that $\mathbb{A}(\bar{X}) \in K_{2n}$. Then there exists $\hat{X} \in \mathbb{S}_+^n$ such that $\mathbb{A}(\hat{X}) \in int(K_{2n})$.

■

Polynomials that are non-negative on the half-line, $\mathbb{R}_+$, or on an interval $[0, 1]$ can be treated similarly.

Trigonometric polynomials

$$p(t) \;=\; \sum_{k=0}^{n} p_k \left(\cos t + \iota \sin t\right),$$

(where $\iota := \sqrt{-1}$ as before) can also be treated similarly.

A central fact related to the above theorems (and their proofs) is that a polynomial is nonnegative on the whole of $\mathbb{R}$ iff the polynomial can be expressed as a sum of squares of polynomials. These types of results go back at least a hundred years.

It has been well known that a polynomial p, with coefficients from $\mathbb{R}$, is nonnegative on the whole real line iff there exist polynomials p_1 and p_2 with real coefficients such that

$$p(t) = [p_1(t)]^2 + [p_2(t)]^2 .$$

If we only require that $p(t) \geq 0$, for all $t \in \mathbb{R}_+$, then there exist polynomials p_1, p_2, p_3, and p_4 such that

$$p(t) = [p_1(t)]^2 + [p_2(t)]^2 + t\left([p_3(t)]^2 + [p_4(t)]^2\right).$$

A related, interesting question goes back to Hermite (in 1894). He asked whether every polynomial p of degree at most n, with the property

$$p(t) > 0, \qquad \forall t \in (-1, 1),$$

can be expressed as

$$p(t) = \sum_{i,j \geq 0:\, i+j \leq n} a_{ij}(1-t)^i(1+t)^j,$$

where $a_{ij} \geq 0$. It was quickly answered "no." However, Hausdorff (in 1921) proved that if the restriction $i + j \leq n$ on the maximum degree of the representing polynomial is relaxed then the answer is "yes." That is, there exist $a_{ij} \geq 0$, for all i, j such that

$$p(t) = \sum_{i,j \geq 0} a_{ij}(1-t)^i(1+t)^j.$$

2.2 Financial Applications

Bertsimas and Sethuraman in [2] discussed the application of SDP representation in financial mathematics. Black-Scholes formula is a popular approach in pricing derivatives under the assumption that the underlying asset follows a *Geometric Brownian Motion* and there exists no arbitrage profit [4].

Let S be the price of the underlying asset. One typical kind of random walk that S follows is Geometric Brownian Motion, which can be expressed as:

$$\frac{dS}{S} = \mu dt + \sigma\phi\sqrt{dt},$$

where μ is the drift rate, σ is the volatility, t is parameter of time and $\phi \sim \aleph(0, 1)$. Moreover, there are no arbitrage opportunities when all risk-free portfolios earn the risk-free rate of return. By constructing a hedging portfolio that consists of $\frac{\partial V}{\partial S}$ shares of an asset and -1 share of an European call option, the Black-Scholes differential equation can be derived as:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV,$$

where V is the price of the option and r is the risk-free interest rate.

The Black- Scholes formula was extremely influential in forming succeeful research in the area, as well as in operation practice of the financial companies in the markets, see for instance [3][8]. As a result, many questions arose. Suppose we do not assume the motion of the underlying asset, only some moments of the price of the asset are given, such as expected price and variance. Can we get a reliable value or bound for the price of derivatives using only no-arbitrage theory?

In order to apply the no-arbitrage theory, Cox and Ross in [5] showed that it is equivalent to the existence of a risk-free probability measure π for the price of the underlying asset. Under this measure, we can define and calculate moments of asset price X. Based on the definition and properties of moments, we can formulate the optimization model for financial problems. The model is solvable efficiently together with the developments in algorithms for SDP, using the theorem of SDP representation in Section 2.1 .

One example is to maximize the expected payoff of a European call option given n moments $(q_1, q_2, \ldots, q_n)$ for the price of the asset. Let q_0=1, the model can be expressed as:

$$\begin{aligned} maximize \quad & E_\pi[max(0, X-k)] = \int_0^\infty max(0, t-k)\pi(t)dt \\ s.t. \quad & E_\pi[X^i] = \int_0^\infty t^i \pi(t)dt = q_i, \; i = 0, 1, 2, \ldots, n \\ & \pi(t) \geq 0, \end{aligned}$$

where k is the strike price for the call option, X is the spot price for underlying asset on maturity, and $max(0, X-k)$ is the payoff for the call option.

The dual of the problem is:

$$\begin{aligned} minimize \quad & \sum_{i=0}^{n} y_i q_i \\ s.t. \quad & \sum_{i=0}^{n} y_i t^i \geq max(0, X-k), \; \forall t \geq 0. \end{aligned}$$

Note that the constraints of dual problem can be expressed as cones of non-negative polynomials, by Theorem 2.1.2 discussed before, the above optimization problem can be formulated as an instance of SDP.

Financial application area is one important aspect for the use of SDP representation methods. We can also use those techniques talked in last section to efficiently model many other problems, which involves non-negative polynomials.

Chapter 3

Portfolio Selection Model

3.1 Model for Single Period

Considering an investment on different types of portfolio, we want to maximize the expected return, taking transaction costs into account, and subject to several kinds of constraints for feasibility.

The single-period portfolio selection model below was introduced by Lobo Fazel and Boyd [7]. The current holdings in n assets are $w := (w_1, \ldots, w_n)^T$. The amounts (in units of asset, not dollars) transacted in these assets are given by the vector $x := (x_1, \ldots, x_n)^T$. After the transactions, the new holdings in the portfolio is $(w + x)$. Let $\phi(x)$ denote the sum of all transaction costs. The problem can be expressed as [7]:

$$\begin{array}{rl} \max & \bar{a}^T(w+x) \\ s.t. & p^T x + \phi(x) \leq \xi \\ & (w+x) \in \mathbf{S}, \end{array}$$

where $\bar{a}$ is the vector of expected returns on each asset, p is the price for assets at the beginning of the period, ξ is the cash amount invested in this period. Then $p^T x$, the investment needed for x, plus the transaction costs $\phi(x)$, must be less than or equal to the budget ξ. $\mathbf{S}$ is the set of feasible portfolios. We will discuss a variety of transaction cost functions and portfolio constraints later.

Assume that transaction costs can be separated, i.e. $\phi(x)$ is the sum of the transaction cost associated with each asset, $\phi(x) = \sum_{i=1}^{n} \phi_i(x_i)$, where ϕ_i is the transaction cost function for asset i.

There are several types of functions $\phi_i(x_i)$ for real world applications, and we will focus on the linear transaction cost functions, such as

$$\phi_i(x_i) = \begin{cases} a_i^+ x_i & x_i \geq 0 \\ -a_i^- x_i & x_i \leq 0 \end{cases} \text{ or } \phi_i(x_i) = \begin{cases} 0 & x_i = 0 \\ \beta_i^+ + a_i^+ x_i & x_i > 0 \\ \beta_i^- - a_i^- x_i & x_i < 0, \end{cases}$$

where a_i^+, a_i^- are the different transaction costs associated with buying and selling asset i, β_i^+ and β_i^- are fixed part of transaction costs.

Define $x_i^+ = x_i$ if $x_i \geq 0$, $x_i^- = x_i$ if $x_i \leq 0$ to express the amount of buying and selling of the asset i, then $x_i = x_i^+ - x_i^-$.

In practice, the transaction cost constraints are frequently not convex. However, we can use convex relaxations to approximate them.

We can also add one asset w_{n+1} to express the holding of cash on hand and x_{n+1} is the cash transacted during this period to involve the cash invested in this period. Then the above problem becomes:

$$\begin{array}{rl} \max & \bar{a}^T(w + x) \\ s.t. & p^T x + \phi(x) \leq \xi + w_{n+1} \\ & (w + x) \in \mathbf{S}, \end{array}$$

where $\bar{a}_{n+1} = 1$, $p_{n+1} = 1$ and $\phi_{n+1}(x_{n+1}) = 0$.

The feasible set of portfolios $\mathbf{S}$ can be discussed in different ways. We will focus on the expression through convex constraints. With convexity, the underlying optimization problems can be solved efficiently by special software based on interior-point methods.

Diversification constraints limit the amount invested in each asset i. Suppose we require that no more than a fraction γ of cash can be invested in r or less assets. We want to avoid concentrating our investments into a small subset of assets to hedge against investment risk. This constraint can be modelled as:

$$\sum_{i=1}^{r} (p \odot x)_{[i]} \leq \gamma p^T x,$$

where $p \odot x := \begin{pmatrix} p_1 x_1 \\ p_2 x_2 \\ \vdots \\ p_n x_n \end{pmatrix}$, $x_{[i]}$ denotes the ith largest component of x, so $x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[n]}$.

An alternative way to express the ith largest component is through introducing new variable $y \in \mathbb{R}^n, t \in \mathbb{R}$.

$$\begin{aligned} \gamma p^T x &\geq rt + e^T y \\ t + y_i &\geq p_i x_i \;, \quad && \forall\, i \in \{1, \ldots, n\} \\ y_i &\geq 0 \;, && \forall\, i \in \{1, \ldots, n\}, \end{aligned} \tag{3.1}$$

where $e \in \mathbb{R}^n$ denote the vector of all ones .

If the constraints (3.1) are satisfied, $\gamma p^T x \geq rt + e^T y$, and $rt + e^T y$ is greater than the sum of any r items of $t + y_i$. Combining with $t + y_i \geq p_i x_i$, it is also greater than or equal to any r items of $p \odot x$, so that $\gamma p^T x$ is greater than or equal to the sum of the r largest components of $p \odot x$, which is what we need. In fact, we can use LP duality Relation to show the equivalence of the two expression above.

Short selling constraints limit the maximum amount of short selling allowed on asset i.

$$w_i + x_i \geq -s_i \;, \quad \forall\, i \in \{1, \ldots, n\},$$

for $s_i \geq 0$.

Variance constraints are based upon the covariance matrix Σ (different notation from before). The covariance matrix is calculated based on the historical data.

Note that the value of holdings at the end of the period is $W = a^T(w + x)$. Assume the return vector a has a Gaussian distribution

$$a \sim \aleph(\bar{a}, \Sigma).$$

The value of holdings is also a random vector

$$W = a^T(w + x) \sim \aleph(\mu, \sigma^2),$$

where $\mu = \bar{a}^T(w + x)$ and $\sigma^2 = E(W - EW)^2 = (w + x)^T \Sigma (w + x)$.

Note that $\bar{a}$ and Σ can be estimated by the mean vector and the covariance matrix of historical data [7]. If covariance terms are estimated independently and the covariance matrix $\tilde{\Sigma}$ is not positive semidefinite, we can compute $\Sigma \in \mathbb{S}^n_+$, such that $\|\Sigma - \tilde{\Sigma}\|$ is minimized, to estimate the covariance matrix.

One method is to computing the eigenvalue decomposition of $\tilde{\Sigma}$:

$$\tilde{\Sigma} = \sum_{i=1}^{n} \lambda_i q_i q_i^T,$$

where λ_i is an eigenvalue of $\tilde{\Sigma}$ and $q_i \in \mathbb{R}^n$ is the corresponding eigenvector.

$$\Sigma := \sum_{i:\lambda_i \geq 0} \lambda_i q_i q_i^T \in \mathbb{S}^n_+,$$

is a positive semidefinite estimation of the variance matrix.

Denoting the maximum standard deviation of wealth W by a parameter σ_{max}, we express the variance constraints as:

$$(w+x)^T \Sigma (w+x) \leq \sigma_{max}^2 \Longleftrightarrow \| \Sigma^{1/2}(w+x) \| \leq \sigma_{max}.$$

This constraint can be expressed as a second-order cone constraint

$$\begin{pmatrix} \sigma_{max} \\ \Sigma^{1/2}(w+x) \end{pmatrix} \in SOC^n.$$

Instead of the matrix square root $\Sigma^{1/2}$ of Σ, we can also use the Cholesky factor G^T of Σ, where G is the unique lower triangular matrix such that $GG^T = \Sigma$. The speed of the calculation of the Cholesky decomposition in practice is much faster than the calculation of the square root, even though they are both $O(n^3)$ in theory. Moreover, the computation of the Cholesky factor seems more numerically stable than the computation of the square root. Then the constraints can be expressed as:

$$\| G^T(w+x) \| \leq \sigma_{max} \Longleftrightarrow \begin{pmatrix} \sigma_{max} \\ G^T(w+x) \end{pmatrix} \in SOC^n.$$

Short risk constraints. These constraints are dealing with VaR (value at risk) for the wealth. We want to require that the wealth W at the end of the period be larger than W^{low} under a probability exceeding η:

$$Prob(W \geq W^{low}) \geq \eta.$$

Note that $W = a^T(w+x) \sim \aleph(\mu, \sigma^2)$, and let $\Phi(z)$ denote the cumulative distribution function of a zero mean, unit variance Gaussian variable.

$$\begin{aligned} Prob\left(\frac{W-\mu}{\sigma} \leq \frac{W^{low}-\mu}{\sigma}\right) \leq (1-\eta) \quad &\Longrightarrow \quad \frac{W^{low}-\mu}{\sigma} \leq \Phi^{-1}(1-\eta) = -\Phi^{-1}(\eta) \\ &\Longrightarrow \quad \mu - W^{low} \geq \Phi^{-1}(\eta)\sigma. \end{aligned}$$

Combining with $\mu = \bar{a}^T(w+x)$ and $\sigma^2 = (w+x)^T\Sigma(w+x)$, we obtain

$$\Phi^{-1}(\eta) \| \Sigma^{1/2}(w+x) \| \leq \bar{a}^T(w+x) - W^{low} \Longleftrightarrow \begin{pmatrix} (\bar{a}^T(w+x) - W^{low})/\Phi^{-1}(\eta) \\ \Sigma^{1/2}(w+x) \end{pmatrix} \in SOC^n.$$

Using Cholesky decomposition, the constraints can be expressed as

$$\Phi^{-1}(\eta) \parallel G^T(w+x) \parallel \leq \bar{a}^T(w+x) - W^{low} \Longleftrightarrow \begin{pmatrix} (\bar{a}^T(w+x) - W^{low})/\Phi^{-1}(\eta) \\ G^T(w+x) \end{pmatrix} \in SOC^n.$$

Note that the formula for variance constraints and short risk constraints are similar, both of which involve the norm of $\Sigma^{1/2}(w+x)$. The short risk constraints not only limit the variance of the wealth, but also has limited relationship between the mean and variance of the wealth.

3.2 Portfolio Selection Model for Multi-Period with Scheduled Cash Infusions

There are many situations in practice when we need to discuss the selection of portfolios under a multi-period model, which means that we will deal with a long time in the future, and divide it into several periods, such as 12 months in one year. During each period, the investor might have scheduled income to be invested on assets. Based upon the partial information about the future periods, we can consider certain utility function for the whole planning horizon, such as expected value. In each period, the assets transacted must be subject to the constraints on the feasible portfolios.

Based upon the previous discussion about the single-period model, we can design similar constraints for the new model. We will deal with the multi-period model using m separate single periods, with different mean vectors and covariance matrices for each period. We require that the amount of assets at the end of each period also be feasible, thus satisfying the constraints of transaction costs, diversification, etc.

We can define $\hat{x} \in \mathbb{R}^{n \times m} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m)$ to be our variable for the new model, where the vector $\hat{x}_i$ denotes the amount of assets transacted during the ith period. In order to express different amounts for buying and selling assets, we can transform the variable space to $\mathbb{R}^{2mn}$, so that the vector $\hat{x}_i^+$ corresponds to the amount of assets bought in period i and the vector $\hat{x}_i^-$ corresponds to the amount of asset sold in period i. The variable can be expressed as:

$$\hat{x} = \begin{pmatrix} \hat{x}_1^+ \\ \hat{x}_2^+ \\ \vdots \\ \hat{x}_m^+ \\ \hat{x}_1^- \\ \vdots \\ \hat{x}_m^- \end{pmatrix} \in \mathbb{R}^{2mn} \geq 0.$$

Define $x_i := \hat{x}_i^+ - \hat{x}_i^-$ to be the transaction amount during the ith period. $y_j := (\sum_{i=1}^{j} x_i) + w$ is the holdings of assets at the end of period j.

In order to express this relationship in matrix notation, we can define a transfer matrix T_j such that $y_j = T_j\hat{x} + w$, where $T_j \in \mathbb{R}^{n\times(2mn)}$,

$$T_j := \left(\begin{array}{cccccccccccccccccccccc} \underbrace{\begin{matrix} 1 & 0 & \dots & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \cdots & 0 & 0 & \dots & 1 \end{matrix}}_{j} & \begin{matrix} 0 & \dots \\ 0 & \dots \\ \vdots & \ddots \\ 0 & \dots \end{matrix} & \underbrace{\begin{matrix} -1 & 0 & \dots & 0 & -1 & \dots & 0 \\ 0 & -1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & -1 & 0 & \dots & -1 \end{matrix}}_{j} & \begin{matrix} 0 & \dots \\ 0 & \dots \\ \vdots & \ddots \\ 0 & \dots \end{matrix} \end{array} \right).$$

Objective function. Our objective is to maximize the expected return at the end of the whole planning horizon

$$maximize \quad \bar{a}_m^T y_m \iff \bar{a}_m^T(w + \textstyle\sum_{i=1}^{m} x_i),$$

where $\bar{a}_m$ is the expected return on each asset for period m. w is the vector of holdings in each asset at the beginning of the planning horizon. One method to estimate $\bar{a}_m$ is by calculating the mean vector of historical assets price for period m.

Transaction cost constraints. Using the same notation as in Section 3.1

$$p_j^T x_j + \phi(x_j) \leq \xi_j \ , \quad \forall j \in \{1, \dots, m\},$$

where $\phi(x_j)$ is under the same definition as in the single-period model, and we still focus on the linear form of transaction costs. ξ_j is the scheduled investment for the period j. Note that we can use real asset price p_1 for the first period, and $p_2, \dots, p_m$ can be estimated by the mean historic prices for periods $2, \dots, m$, which are $\bar{a}_1, \ldots, \bar{a}_{m-1}$. This selection of p can make sure that x_1 is strictly feasible, and $x_2, \ldots, x_m$ are feasible based upon estimation.

We can also add one asset to express the holding of cash on hand, denoted it by ζ_i for $i \in \{1, \ldots, m\}$. The above problem becomes:

$$p_j^T x_j + \phi(x_j) + \zeta_j \leq \xi_j + \zeta_{j-1} \ , \quad \forall j \in \{1, \ldots, m\},$$

Diversification constraints. Using the constraint discussed in the single-model:

$$\textstyle\sum_{i=1}^{r} (p_j \odot x_j)_{[i]} \leq \gamma p_j^T (x_j) \ , \quad \forall j \in \{1, \ldots, m\}.$$

The alternative way to express the ith largest component through introducing new variable $Y \in \mathbb{R}^n, t \in \mathbb{R}$ can also be applied here, as in equations (3.1) on page 22.

Short selling constraints. $s \in \mathbb{R}_+^n$ is the vector of lower bound (which could also represent a credit line):

$$y_j \geq -s \ , \quad \forall j \in \{1, \ldots, m\}.$$

Variance constraints

$$\| \Sigma_j^{1/2} (y_j) \| \leq (\sigma_{max})_j \ , \quad \forall j \in \{1, \ldots, m\}.$$

Note that when we begin to forecast the future return, the variance matrix Σ_j and mean vector $\bar{a}_j$ are calculated from previous data for each period, and they can only reflect partial information. Throughout the course of the horizon, we can update them to include the new information and make the forecast more suitable to the real case. Also we can use different variance matrix and mean for each period, this kind of estimation can include the monthly or seasonal changes on the values of the assets in the portfolio.

Short risk constraints. Based on the same assumption as the single-period model, the short risk constraint can be represented as:

$$\Phi^{-1}(\eta) \| \Sigma_j^{1/2} (y_j) \| \leq \bar{a}_j^T (y_j) - W^{low} \ , \quad \forall j \in \{1, \ldots, m\}.$$

The previous discussion on the Cholesky decomposition of the variance matrix is also applicable here. Moreover, the corresponding constraints are also representable as second-order cone constraints.

We note that our models presented in this chapter can be extended using the robust optimization approach. The resulting optimization problems are still SOCPs.

Chapter 4

Computational Results

4.1 Data Description

The historical data for experiment is obtained from database of CRSP (The Center for Research in Security Prices), which creates and maintains premier historical US databases for publicly traded stocks (NASDAQ, AMEX, NYSE), indices, bond, and mutual fund securities. The database used is maintained and supported by SOAR (School of Accountancy Research) at the University of Waterloo.

4.2 Software Package

We use Sedumi 1.05, an add-on toolbox for MATLAB. It implements the *self-dual* embedding technique for optimization problems over self-dual and homogeneous (which is equal to symmetric) cones[15]. As the special and common case of symmetric cone, SDP and SOCP can be efficiently solved by Sedumi.

The feature that Sedumi supports complex value data is also an attraction for our choice to solve the portfolio selection models, because the estimation for the variance matrix from real data may be indefinite. Therefore, when we use the matrix $V^{\frac{1}{2}}$, complex values can occur.

The version of MATLAB is 6.5 under the workstation of $Windows^{®}$ XP and *UNIX*. The machine for experiment is powered by processor Intel $Pentium^{®}$ M 1.3 G, with 256MB DDR SDRAM. The server for *UNIX* is cpu101.math, which is a general math faculty CPU server and the CPU is UltraSPARC IIe with speed 648 MHz and memory 1.5 GB.

4.3 Experiments

We design series of experiments to examine the efficiency and calculation capacity for the portfolio selection model.

First, we want to compare the optimal strategy obtained from single-period model and multi-period model with scheduled cash infusion in each period of the whole horizon. Second, we want to keep the size of the whole horizon and increase the number of assets in the portfolio to find the maximum number of assets that the model can solve. Third, we want to keep the number of assets in the portfolio and increase the size of the whole horizon for forecast to find the maximum length that the model can solve.

For stocks on 20 companies with Nasdaq company number between 60006000 and 60006200, such as *RAYMOND, GRIFFON, KEY TRON, LIFELINE, MAIR HOL, UNIFIRST, LAFARGE, BURLINGT*, we selected the monthly average price between Jan $1st$, 1993 and Dec $31th$, 2003. We use data from Jan $1st$, 1993 and Dec $31th$, 2002 to forecast the optimal investment strategy for the whole year of 2003 under scheduled cash flow for each month in 2003, and the objective is to maximize the total profit earned at the end of 2003.

Note that calculating the average price matrix and covariance matrix over ten years for each different month, can reflect the differences between the periods we want to forecast. These matrices are shown in the appendix.

The process for the experiment can be stated as:

1. Estimate mean vectors and covariance matrices $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{12}, \Sigma_1, \Sigma_2, \ldots, \Sigma_{12}$ from the historical data.
2.
    - Use $\bar{a}_1, \Sigma_1$, under single-period model to find optimal decision x^1 for the $1st$ period
    - Use $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{12}, \Sigma_1, \Sigma_2, \ldots, \Sigma_{12}$, under multi-period model to find optimal decision $\tilde{x}^1$ for the $1st$ period.
3. Apply x^1 and $\tilde{x}^1$ to the $1st$ month and go to step 2 to forecast the optimal strategy beginning with next month.

Repeat the process above until the end of the 12th month in 2003.

Note that we use different mean vectors and covariance matrices in the forecasting process. If we do not need to reflect the difference between the periods, which is using the same mean vector and covariance matrix for the whole 12 months, we can use the newest asset price to update the mean vector and covariance matrix in step 3 to get reliable forecast.

For transactions cost and constraint parameters, we choose:

$$a_i^+ = 3.5 \quad a_i^- = 2 \quad \beta_i^+ = \beta_i^- = 0 \ , \quad \forall i \in \{1, \ldots, 20\}.$$

For the Diversification constraints, we used the formulation (3.1) on page 22. The parameters in (3.1) and short selling constraints are:

$$r = 3 \quad \gamma = 70\% \quad s_i = 0 \ , \quad \forall i \in \{1, \ldots, 20\}.$$

For the variance constraints and two different short risk constraints, we select:

$$\sigma_{max} = \sqrt{1500} \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 80\% \\ 95\% \end{pmatrix} \quad W^{low} = \begin{pmatrix} 50 \\ 25 \end{pmatrix}.$$

We carry out the process described above and all the parameters stay the same for each period in the experiment. The first experiment is with cash flow 200 for each month and initial amount $w = 0$, and the second experiment is with cash flow 100 for each month and initial amount $w = 0$.

Adding another asset to express the cash holdings for each period, the third experiment follows the same process discussed above. The cash flow is 50 for each month while parameters stay the same except $W^{low} = \begin{pmatrix} 0 \\ -10 \end{pmatrix}$ and $\sigma_{max} = \sqrt{1000}$.

We increase the size of the problem as discussed before, still following the same process for experiment.

4.4 Numerical Result and Analysis

The optimal strategy and holdings for the first three experiments are listed below:

Result for experiment with cash flow 200		
Single period model	value of the commodity holdings	1044.9
	total transaction costs	414.8612
	total cash and commodity holdings	1044.9+12×200-414.8612 =3030.04
Multi-period model	value of the commodity holdings	1131.4
	total transaction costs	371.8835
	total cash and commodity holdings	1131.4+12×200-371.8835=3159.51

Result for experiment with cash flow 100		
Single period model	value of the commodity holdings	722.2259
	total transaction costs	289.8983
	total cash and commodity holdings	722.2259+12×100-289.8983 =1632.36
Multi-period model	value of the commodity holdings	852.7675
	total transaction costs	281.7293
	total cash and commodity holdings	852.7675+12×100-281.7293=1771.04

Result for experiment with cash flow 50 and new variable for cash on hand		
Single period model	value of the commodity holdings	377.7366
	cash holdings	208.2462
	total cash and commodity holdings	585.9828
Multi-period model	value of the commodity holdings	231.3497
	cash holdings	415.2706
	total cash and commodity holdings	646.6203

The optimal investment strategy for each period is shown in the appendix. Based upon the numerical results, we can find the advantage of multi-period model. The single period model is more 'greedy', and the multi-period model considers further future time when 'forecasting'. The multi-period model can provide better investment strategy at the end of the planning horizon. Note that 'better' means bigger value of holdings at the end and smaller total transaction costs.

In order to test the maximum number of assets in the portfolio we can solve for the multi-period model. We select another data of stocks on 100 companies with Nasdaq company number between 60007000 AND 60008000. The number of assets are increasing from 20 to 60, and the size of the optimization problem for multi-period model is 1573 variables with 5293 constraints.

The last series of experiments are focusing on finding the maximum length of horizon we can solve for the multi-period model. We still use the data with 20 stocks and the same parameters as above, but change the length of historical part and forecast part of the data. Increase the number of periods from 12 to 24,. . . .

Under Windows environment, we encountered error report from MATLAB when we wanted to forecast for 41 months. The optimization problem for 40 months uses 1681 variables and 5860 constraints. Under Unix environment, report of out of memory from MATLAB came out when we wanted to forecast for 37 months. The size of multi-period model for 36 months is 1517 variables and 5281 constraints.

These differences for capacity are arising from the size of physical memory under different computing environments. It is not stable because of the status of memory on different machines.

One result we get is for the experiment with 60 assets for 12 months. The optimization problem consisted of 1573 variables and 5293 constraints. We used the same parameters as above except:

$$\sigma_{max} = \sqrt{5000},\ W^{low} = \begin{pmatrix} -50 \\ -100 \end{pmatrix},$$

and the cash infusion for each month is 300. The result was obtained nearly after 1 hour of computing. The largest programming took 35 iterations with 535.8 seconds computing time to slove.

Result for experiment with 60 assets		
Single period model	value of the commodity holdings	4597.8
	cash holdings	300
	total cash and commodity holdings	4897.8
Multi-period model	value of the commodity holdings	5309.1
	cash holdings	0
	total cash and commodity holdings	5309.1

When changing parameters for feasible portfolio, some instances of the single period model was infeasible while multi- period model never encountered this difficulty. That is because of the greedy approach of the single period model, it is easier to reach the boundary of the feasible set and therefore become less stable. This happened for the experiment with 20 assets and 36 months forecasting, with the same parameters as above.

Appendix A

Below, there are the mean vectors and variance matrices for historical prices of 20 stocks. They are calculated by the historical data from Jan 1st, 1993 to Dec 31th, 2002, which are used in the experiment 1 with cash flow 200, experiment 2 with cash flow 100, experiment 3 with cash flow 50 and new variable for cash on hand.

The mean vector are listed for different months and only covariance matrix of Jan is listed because of limited space.

Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
3.8093	3.9081	1.917	4.048	3.175	4.2274	1.6404	3.7175	3.7803	2.9396	4.0765	4.1174
10.1825	9.8812	7.3388	9.458	8.92	8.696	7.2108	7.7712	9.1759	5.5695	8.6662	8.7862
10.0325	8.7928	11.889	7.7213	11.4662	11.3178	5.8213	6.5275	9.395	8.8317	11.5588	11.9453
6.6948	3.1522	6.618	5.2017	6.0515	4.7458	3.6989	3.2375	5.3924	3.1199	2.2519	4.2429
3.899	-2.7853	4.524	5.035	1.55	1.2825	0.6569	3.2	3.4395	1.4855	-1.3194	3.2197
4.4531	4.5412	4.2434	4.1997	4.0285	3.18	2.8751	3.6876	3.5119	2.7493	3.5931	3.3829
1.7556	1.0164	1.0053	0.6925	-0.0012	1.9203	1.25	1.4784	0.1223	0.4786	0.9772	1.6539
10.9647	7.9025	8.4478	12.174	11.2122	10.7303	9.9407	9.599	10.3097	10.9052	13.6838	11.4487
10.3033	10.8037	10.0588	10.0188	10.0338	9.96	9.615	9.2525	9.8288	9.81	10.175	10.8058
6.144	5.986	5.9383	5.7858	6.1625	5.7945	6.0375	5.8232	6.1502	5.6172	6.015	5.2022
14.525	14.01	14.0063	10.482	13.9952	14.61	15.0125	14.559	12.1813	15.6075	12.185	16.411
19.211	19.1498	19.325	19.5123	13.8313	19.38	19.4175	12.9692	13.0303	19.154	16.9593	19.8875
26.2022	25.684	26.183	27.9842	27.4983	26.341	26.464	26.1812	25.3807	25.3475	25.8793	27.1108
15.5195	16.644	17.7388	18.3623	17.2898	17.4875	17.3238	17.9125	16.8788	15.5967	15.4308	15.8313
25.7188	25.4275	25.5718	23.929	24.1725	24.3133	23.66	23.7957	25.3595	25.5405	25.4675	26.1413
5.729	4.721	2.3175	3.9	4.3145	4.3877	4.3812	3.0637	1.7456	4.1101	2.6408	5.4292
9.6537	9.7737	8.7751	8.6435	9.346	9.5825	9.4289	9.3007	9.5723	9.5892	9.6573	9.2977
9.7075	9.7532	10	9.9675	5.5638	8.8255	9.9983	9.8755	7.226	10.0095	9.9095	10.1203
9.156	9.6928	9.8938	9.925	10.1855	9.5648	9.2508	8.7706	8.6734	8.6516	8.4525	9.065
5.8532	6.7594	5.4562	5.6535	5.4465	6.5856	5.9556	6.3143	7.1507	6.3812	5.5174	6.3071

Covariance matrix for January

7.3711	2.2715	3.4555	1.9606	4.2644	-0.7312	2.2549	1.2495	-0.5727	5.7359	4.5924	3.1451	6.0231	3.5278	7.4589	0.4991	0.3551	3.0285	-0.1846	5.1559
2.2715	29.3192	11.1474	3.0409	-0.7437	-3.5204	1.5954	56.0378	3.6796	-6.4441	12.9727	-2.4021	-3.5953	-16.149	7.0995	-1.2498	15.9639	-0.8884	-11.4212	7.1982
3.4555	11.1474	48.8895	-1.6298	10.8643	-6.334	-3.9121	-2.9144	13.2984	-20.2356	48.705	-10.8439	46.3605	-16.4018	15.4603	-5.2358	10.0472	9.4785	-14.8952	11.5962
1.9606	3.0409	-1.6298	4.2749	-3.2819	-0.0079	-3.7969	0.9414	-1.4587	-0.2151	-6.5107	-4.125	-9.7154	-2.1704	0.4782	-0.8228	3.7274	-0.0287	-1.5546	2.7827
4.2644	-0.7437	10.8643	-3.2819	24.7693	0.5311	7.6513	-5.5812	-2.9485	-7.2036	13.54	-3.4328	18.2761	-4.3171	-0.3843	0.1232	-0.6584	4.0135	-4.0064	4.1799
0.7312	-3.5204	-6.334	-0.0079	0.5311	5.6379	-0.1781	-1.4637	-3.6939	0.1003	-10.8207	-3.3516	-6.2061	6.2172	7.247	0.8625	-4.6663	-0.0526	2.1141	3.4414
-2.2549	1.5954	-3.9121	-3.7969	7.6513	-0.1781	9.1764	9.3254	0.0519	2.0432	4.0097	6.8309	2.7774	-0.2186	0.5027	2.4709	0.3758	-0.3452	1.5204	-3.6169
1.2495	56.0378	-2.9144	0.9414	-5.5812	-1.4637	9.3254	155.808	5.6714	-1.9038	3.1838	6.1377	-28.4137	-29.5018	0.2517	1.6384	16.9093	-14.149	-9.886	4.0696
0.5727	3.6796	13.2984	-1.4587	-2.9485	-3.6939	0.0519	5.6714	11.3644	-4.7579	22.184	10.0299	17.1058	-2.4439	12.4538	0.7635	3.4841	2.5282	0.3905	-0.2072
-5.7359	-6.4441	-20.2356	-0.2151	-7.2036	0.1003	2.0432	-1.9038	-4.7579	12.0209	-19.9638	3.4035	-20.9859	2.273	-15.4693	1.3105	-3.7459	-6.1068	6.027	-9.3901
4.5924	12.9727	48.705	-6.5107	13.54	-10.8207	4.0097	3.1838	22.184	-19.9638	75.8326	18.1179	63.981	-11.3044	29.7888	-0.841	12.7285	10.9253	-11.2832	7.4857
3.1451	-2.4021	-10.8439	-4.125	-3.4328	-3.3516	6.8309	6.1377	10.0299	3.4035	18.1179	37.8071	11.6372	13.7497	14.9097	6.7635	-1.7053	0.575	11.224	-7.893
6.0231	-3.5953	46.3605	-9.7154	18.2761	-6.2061	2.7774	-28.4137	17.1058	-20.9859	63.981	11.6372	74.7185	3.855	22.6082	-0.7249	-1.0414	14.491	-5.1101	5.1378
3.5278	-16.149	-16.4018	-2.1704	-4.3171	6.2172	-0.2186	-29.5018	-2.4439	2.273	-11.3044	13.7497	3.855	27.9803	13.9678	4.3751	-11.9177	4.2537	11.6559	-1.6197
7.4589	7.0995	15.4603	0.4782	-0.3843	7.247	0.5027	0.2517	12.4538	-15.4693	29.7888	14.9097	22.6082	13.9678	69.354	4.3248	7.2704	11.9735	-1.6146	19.611
0.4991	-1.2498	-5.2358	-0.8228	0.1232	0.8625	2.4709	1.6384	0.7635	1.3105	-0.841	6.7635	-0.7249	4.3751	4.3248	1.9694	-1.2522	0.1345	3.3597	-1.4385
0.3551	15.9639	10.0472	3.7274	-0.6584	-4.6663	0.3758	16.9093	3.4841	-3.7459	12.7285	-1.7053	-1.0414	-11.9177	7.2704	-1.2522	14.8911	1.8846	-8.7397	4.6079
3.0285	-0.8884	9.4785	-0.0287	4.0135	-0.0526	-0.3452	-14.149	2.5282	-6.1068	10.9253	0.575	14.491	4.2537	11.9735	0.1345	1.8846	5.9745	-1.5555	4.1713
-0.1846	-11.4212	-14.8952	-1.5546	-4.0064	2.1141	1.5204	-9.886	0.3905	6.027	-11.2832	11.224	-5.1101	11.6559	-1.6146	3.3597	-8.7397	-1.5555	11.0309	-7.0974
5.1559	7.1982	11.5962	2.7827	4.1799	3.4414	-3.6169	4.0696	-0.2072	-9.3901	7.4857	-7.893	5.1378	-1.6197	19.611	-1.4385	4.6079	4.1713	-7.0974	13.1275

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec	holdings
Transaction amount in each month	0	0	0	15.0726	-4.5018	0	0	0	0	-9.6603	0	0	0.9105
	3.2817	0	1.0893	0.3814	-4.7525	0	0	0	0	0	0	0	0
	0	0.0675	2.5589	0.3645	-2.2198	0	0	0	0	-0.7711	0	0	0
	11.8934	-10.0735	4.4161	-1.3928	2.3755	0	0	0	0	-7.2187	0	0	0
	0	0	1.5273	-1.5273	0	0	0	0	0	2.1149	0	0	2.1149
	2.3999	8.9463	-4.9985	7.15	4.6114	0	0	0	0	-16.8364	0	0	1.2727
	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	1.1806	-1.1806	0	0	0	0	0.3771	0	0	0.3771
	9.3359	6.0977	-0.5015	5.6889	-4.4894	0	0	0	0	15.2831	0	0	31.4147
	0	0	1.938	0.0556	1.7238	0	0	0	0	1.1629	0	0	4.8803
	0.0272	0.8784	2.0689	0.9857	-2.7963	0	0	0	0	0.8428	0	0	2.0067
	1.1142	1.9475	1.3754	0.5243	1.28	0	0	0	0	1.054	0	0	7.2954
	0	1.0178	2.4774	0.9343	2.2968	0	0	0	0	1.9615	0	0	8.6878
	0	0.3551	-0.2756	0.5887	-0.3023	0	0	0	0	1.0091	0	0	1.375
	0	0	0	2.5442	-2.5442	0	0	0	0	0	0	0	0
	2.9349	5.1761	-8.1109	0	5.0754	0	0	0	0	-1.9423	0	0	3.1332
	0	0	0	1.295	-1.295	0	0	0	0	2.717	0	0	2.717
	0	0	0	1.0878	2.5481	0	0	0	0	1.4642	0	0	5.1001
	0	0	0	0	0	0	0	0	0	5.0592	0	0	5.0592
Transaction cost	61.9743	59.046	48.7889	78.6274	63.9041	0	0	0	0	102.5204	0	0	414.8611

Table 1: Numerical result for single period model in experiment 1 with cash flow 200

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec	holdings
Transaction amount in each month	0	1.0244	1.573	-2.502	0	0	0	-0.0765	0.0188	0.8528	0	4.472	5.3625
	3.9306	-0.412	-0.9053	-2.4809	0	0	0	2.1666	-2.299	0.2265	0	0	0.2265
	0	0.7005	0.5566	2.0327	0	0	0	-1.1245	-1.8953	0.012	0	1.7064	1.9884
	0	1.5593	-0.9472	-0.2263	0	0	0	-0.3595	-0.0263	0	0	0	0
	0	0.8178	2.1385	-1.333	0	0	0	0.7065	-2.3298	2.8116	0	1.8164	4.628
	0	2.5964	1.61	0.2002	0	0	0	4.759	-9.1657	0.3777	0	-0.3777	0
	0	0.9552	0.361	3.3745	0	0	0	-4.649	-0.0417	0	0	0	0
	0	0.608	1.2235	-1.3069	0	0	0	-0.5168	-0.0078	0	0	0.8071	0.8071
	0	0.4514	0.0496	-0.2243	0	0	0	0.6194	0.0172	1.5004	0	-2.2629	0.1508
	8.5327	1.6087	-0.6899	-2.7007	0	0	0	7.8024	-0.7068	11.2282	0	9.9427	35.0173
	0	0.6198	0.9749	0.7955	0	0	0	-1.357	0.9701	0.8544	0	0.7411	3.5988
	0.6776	-0.1775	-0.3425	-0.0933	0	0	0	1.148	1.149	-2.3612	0	0	0
	0.9972	1.0657	1.0691	1.0324	0	0	0	0.8831	0.8641	0.7744	0	0.6808	7.3668
	1.7165	1.7105	1.9257	1.8375	0	0	0	1.6051	0.0971	1.4411	0	1.2293	11.5628
	0.9462	-0.5093	-0.3453	-0.0313	0	0	0	-0.0273	-0.033	0.1391	0	-0.0659	0.0732
	0	0.6265	1.167	3.4999	0	0	0	1.31	1.8056	-8.4089	0	2.1436	2.1437
	1.8196	-0.2807	-1.2575	-0.0373	0	0	0	-0.0751	0.7758	2.5398	0	2.3599	5.8445
	0	0.4782	-0.0907	-0.1875	0	0	0	0.9218	0.5127	1.9961	0	-2.4615	1.1691
	0	0.8936	0.8676	1.4417	0	0	0	1.4523	1.3641	1.0758	0	-1.9369	5.1582
	0	1.3209	0.8885	0.2947	0	0	0	-1.7792	4.1671	-0.5568	0	3.5434	7.8786
Transaction cost	37.2408	35.453	33.3883	40.1418	0	0	0	56.7134	39.9695	62.9868	0	65.9899	371.8835

Table 2: Numerical result for multi-period model in experiment 1 with cash flow 200

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec	holdings
Transaction amount in each month	0	0	0	9.8801	-3.0309	0	0	0	0	0	-0.8588	0.9574	6.9478
	4.2043	0	-3.8932	1.5388	-1.85	0	0	0	0	0	2.4303	-2.4303	0
	0	0	1.0462	0.0348	4.1561	0	0	0	0	0	0.837	1.0993	7.1734
	4.8457	-3.5827	8.8431	-5.2805	-1.3347	0	0	0	0	0	2.0541	-5.5449	0
	0	0	0	0.9163	-0.9163	0	0	0	0	0	1.3247	0.8209	2.1456
	0	5.1565	-4.8739	3.0199	5.2928	0	0	0	0	0	-3.3759	-3.4209	1.7985
	0	0	0	0	0	0	0	0	0	0	0	0	0
	0.634	-0.634	0	0.1458	-0.1458	0	0	0	0	0	0	0.181	0.181
	0	0	0	0.4986	1.0492	0	0	0	0	0	0.638	-0.5413	1.6445
	5.8482	4.3152	-1.5015	4.4539	0.5364	0	0	0	0	0	5.6117	7.8121	27.076
	0	0	0.7924	0.1533	0.6043	0	0	0	0	0	-1.5499	0.4774	0.4775
	0	0.0882	0.8459	0.4163	-0.1088	0	0	0	0	0	0.0332	-0.9652	0.3096
	0	0.6791	0.7268	0.2214	0.4487	0	0	0	0	0	0.342	0.4386	2.8566
	0	0.5866	1.0129	0.3946	0.8051	0	0	0	0	0	0.5756	0.7919	4.1667
	0	0.4095	0.6241	0.2486	0.4983	0	0	0	0	0	0.3027	0	2.0832
	0	0	0	0.7835	-0.7835	0	0	0	0	0	0	1.381	1.381
	1.8385	2.9834	-1.441	0	1.6714	0	0	0	0	0	0.4265	-3.1159	2.3629
	0	0	0	0.547	-0.375	0	0	0	0	0	0.8514	0.9749	1.9983
	0	0	0	0.4595	0.8932	0	0	0	0	0	0	0.0464	1.3991
	0	0	0	0	0	0	0	0	0	0	1.3325	2.2828	3.6153
Transaction cost	34.7412	32.6537	39.4924	52.7055	40.456	0	0	0	0	0	39.3037	50.5459	289.8984

Table 3: Numerical result for single period model in experiment 2 with cash flow 100

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec	holdings
Transaction amount in each month	0	0	0	3.7033	0	0	0	-3.7033	0	1.3684	0.3117	2.7594	4.4395
	3.5473	0	-1.4571	0.8637	-2.9539	0	0	0	0	2.2566	2.1761	-2.1579	2.2748
	0	0	0.9041	0	3.3659	0	0	-2.6583	0	-0.8848	0.8175	1.0529	2.5973
	0	4.1873	2.7492	0	-0.0836	0	0	-6.8529	0	0	4.4136	-4.4136	0
	0	0	0	0	0	0	0	1.6852	0	0.0098	0.742	1.1208	3.5578
	0	1.5067	0	0	2.4962	0	0	3.1061	0	0	-0.5845	-6.5245	0
	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0.0365	0.0365
	0	0	0	0	0	0	0	0.885	0	0	0.4055	0	1.2905
	9.0771	2.2042	-0.4781	2.8982	0.6102	0	0	-1.2473	0	5.9907	1.7085	6.8593	27.6228
	0	0	0.6847	0	0.6388	0	0	0.5332	0	0.4401	-1.7145	0.4573	1.0396
	0	0	0.731	0.8643	0.0119	0	0	-0.0534	0	-1.163	-0.3908	0	0
	0.3564	0.4471	0.6456	0.5346	0.4744	0	0	0.4729	0	0.3989	0.334	0.4201	4.084
	0.6522	0.7465	0.8753	0.8193	0.8512	0	0	0.8595	0	0.7423	0.5622	0.7585	6.867
	0.0885	0.322	0.3805	0.5162	0.5268	0	0	0.4559	0	0.2659	0.2956	-0.5947	2.2567
	0	0	0	0	0	0	0	1.5232	0	-1.5232	1.2341	1.3227	2.5568
	0.516	2.8859	-2.0045	0	0.8307	0	0	0.287	0	0.7348	-1.0548	-0.9583	1.2368
	0	0	0	0.1776	-0.1776	0	0	1.3146	0	1.0282	0.8316	0.9338	4.1082
	0	0	0	0.9539	0.9443	0	0	0.7924	0	0.5541	0.4338	-0.9854	2.6931
	0	0	1.1247	0	1.1583	0	0	-1.5875	0	-0.6614	1.6795	2.1864	3.9
Transaction cost	28.4748	24.5997	20.13	22.6621	27.0321	0	0	39.9325	0	31.8121	35.6361	51.45	281.7294

Table 4: Numerical result for multi-period model in experiment 2 with cash flow 100

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec	holdings
	0	0	0	5.0041	-0.4721	0	-1.1624	0	0	0	0	0	3.3696
	3.9242	0.1914	-2.3042	0.877	-2.1048	0.1991	0	0	0	0	0.3462	0.1937	1.3226
	0	0	0.4031	0	1.7726	0	-1.6355	0	0	0	0.1192	0.0643	0.7237
	1.2614	0	3.0512	0	0	0.8494	-1.0465	-4.1155	0	0	0.9354	0.4597	1.3951
	0	0	0	0.5752	0	-0.5752	0	0.3873	-0.3196	0	0	0	0.0677
	0	0.3874	0	0	1.1005	0	0	0.7138	0	0	0	0	2.2017
	0	0	0	0	0	0	0	0	0	0	0	0	0
Transaction	0.225	0	-0.225	0	0	0.1034	-0.1034	0	0	0	0.0737	0.1781	0.2518
amount	0	0	0	0.313	0.3813	0	0.4507	0.2034	0.1041	0	0	0	1.4525
in each month	1.5223	5.2321	0	0.7636	0	0.6072	0	0	0	0	0.7994	0.4572	9.3818
	0	0	0.3053	0	0.2196	0	0.2539	0.1225	0.0643	0	-0.2304	0	0.7352
	0.0874	0.0663	0.3259	0.2614	0.3157	0.0614	0.3293	-0.376	0.0762	0	0	0.0421	1.1897
	0	0.0455	0.3518	0.139	0.1631	0.0351	0.2334	0.1087	0.0573	0	0.0487	0.0257	1.2083
	0	0.0759	0.3903	0.2477	0.2926	0.0648	0.4029	0.1975	0.0944	0	0.082	0.0463	1.8944
	0.0597	0.053	0.2405	0.1561	0.1811	0.0362	0.2182	0.1048	0.0549	0	0.0431	0.0265	1.1741
	0	0	0	0.4918	0	0	0.8337	0.35	0.1941	0	-0.059	0	1.8106
	0.4786	1.3609	0	0	0	0	0	0	-0.6734	0	0.1776	0.089	1.4327
	0	0	0	0.3434	0	0	0.598	0.3021	0.157	0	0.1213	0	1.5218
	0	0	0	0.2884	0.3246	0	0.3788	0.1821	0.0905	0	0	-0.1543	1.1101
	0	0	0	0	0	0.1885	0	0	0.2617	0	0	0.1336	0.5838
Cash on hand	0	4.9327	0.4183	0	0	36.6054	34.8552	50.9175	86.4815	136.4815	168.7525	208.2462	
Transaction cost	26.4548	25.9433	22.7964	33.1126	21.782	8.6585	20.842	18.3359	6.0264	0	10.192	6.315	

Table 5: Numerical result for single period model in experiment 3 with cash flow 50 and cash on hand

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec	holdings
	0	0	0	4.2744	0	0	0	0	0	0	0	0	4.2744
	0.4537	0.3359	0.5523	0.4783	-0.5126	0	0	0	0	0.1939	0.2603	0.1869	1.9487
	0	0	0	0.1413	1.3776	-0.1475	-0.2408	-0.0512	0.4379	0.0666	0.0896	0.0621	1.7356
	0	0	0	0	0	0	0	0	0	0	0	3.1355	3.1355
	0	0	0	0	0.3116	0	0	0	-0.3116	0	0	0	0
	0	0	0	0	0.8448	0	0	0	0	0	0	0	0.8448
	0	0	0	0	0	0	0	0	0	0	0	0	0
Transaction	0.1907	0	0	0.1801	0	0	0	-0.3708	0	0.087	0.24	0.1719	0.4989
amount	0	0	0	0.155	0.1675	0.0532	0.0632	0.0692	0.1128	0	0	0.0483	0.6692
in each month	1.2907	1.1356	2.2803	0.6478	0	0	0	0	0	0.404	0.601	0.4412	6.8006
	0	0	0	0.0979	0.0964	0.0298	0.0356	0.0417	0.0697	0	0	0	0.3711
	0.0212	0	0.1322	0.1294	0.1387	0.0454	0.0462	0.0481	0.0826	0.041	0.0502	0.0406	0.7756
	0	0.0152	0	0.0688	0.0716	0.026	0.0327	0.037	0.0621	0.0279	0.0366	0.0248	0.4027
	0	0	0	0.1227	0.1285	0.048	0.0565	0.0673	0.1023	0.0519	0.0616	0.0447	0.6835
	0	0.0619	0.0279	0.0773	0.0795	0.0268	0.0306	0.0357	0.0594	0.0267	0.0324	0.0255	0.4837
	0	0	0	0.2435	0.2574	0	0.1169	0.1192	0.2103	0	0	0	0.9473
	0.4057	0.451	0.6646	0.331	0.0052	0	0	-0.061	-0.04	0.0914	0.1336	0.0859	2.0674
	0	0	0	0.17	0.1898	0.0653	0.0839	0.1029	0.1701	0.0718	0	0.0282	0.882
	0	0	0	0.1428	0.1426	0	0	0.062	0.098	-0.2212	-0.2242	0	0
	0	0	0	0.3291	0	0	0	0	0	0	0	0.1289	0.458
Cash on hand	35.3166	71.5531	98.1005	112.9697	138.8299	184.0009	227.5552	269.2364	304.5008	346.1853	384.8016	415.2706	
Transaction cost	8.267	6.9984	12.8004	26.5628	14.3644	1.3251	2.1113	3.0068	5.622	4.16	5.7171	15.4861	

Table 6: Numerical result for multi-period model in experiment 3 with cash flow 50 and cash on hand

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