# COMPREHENSIVE EXAM: ENUMERATION
## July 26, 2000, 1-4 p.m.

1. Let $c(n)$ be the number of labelled, rooted trees on vertices $\{1,\ldots,n\}$, and let $C(t) = \sum_{n\geq 1} c(n)\frac{t^n}{n!}$.

(a) Give a decomposition for trees to prove that

$$C = te^C.$$

(b) Deduce from part (a) that

$$t\frac{dC}{dt} = \frac{C}{1-C}.$$

(c) By interpreting $t\frac{dC}{dt}$ as the generating function for labelled, doubly rooted trees, and writing $\frac{C}{1-C}$ as $C + C^2 + \cdots$, give a combinatorial proof of the result in part (b).

(d) Deduce from part (b) and Lagrange's Theorem that, for $n \geq 1$,

$$n^{n-1}(n-1) = \sum_{i=1}^{n} \binom{n}{i} i^{i-1}(n-i)^{n-i}.$$

2. Let $F(x,z) = \prod_{i\geq 1}(1 + xz^{2i-1})$.

(a) Prove that $F(x,z) = (1+xz)F(xz^2,z)$, and hence deduce that

$$\prod_{i\geq 1}(1 + xz^{2i-1}) = 1 + \sum_{m\geq 1} x^m z^{m^2} \prod_{j=1}^{m}(1-z^{2j})^{-1}.$$

(b) Give a combinatorial proof of the infinite product - infinite sum identity in part (a).

3. Let $a(n,k)$ be the number of permutations of $\{1,\ldots,n\}$ in which $k$ of the cycles in the disjoint cycle representation have odd length.

(a) Prove that

$$\sum_{n,k\geq 0} a(n,k)u^k\frac{x^n}{n!} = (1-x)^{\frac{-u-1}{2}}(1+x)^{\frac{u-1}{2}}.$$

(b) Deduce from part (a) that

$$a(2m,0) = \prod_{i=1}^{m}(2i-1)^2.$$

(c) A *matching* on the set $\{1,\ldots,2m\}$ is an unordered collection of $m$ unordered pairs of elements of $\{1,\ldots,2m\}$ so that each element of $\{1,\ldots,2m\}$ occurs in exactly one pair. Let $M(2m)$ be the number of matchings on the set $\{1,\ldots,2m\}$. Prove that

$$M(2m) = \prod_{i=1}^{m}(2i-1).$$

(d) Parts (b) and (c) imply that

$$a(2m,0) = M(2m)^2.$$

Give a direct combinatorial proof of this equality, by finding a bijection between permutations of $\{1,\ldots,2m\}$ in which all cycles have even length, and ordered pairs of matchings on $\{1,\ldots,2m\}$.

4. Let $d(n)$ be the number of lattice paths from $(0,0)$ to $(n,n)$, with steps $(0,1)$ or $(1,0)$, in which no step lies below the line $y = x$, for $n \geq 0$ (for $n = 0$, there is a single such path, with no steps). Let

$$D(x) = \sum_{n\geq 0} d(n)x^n.$$

(a) Prove that

$$D = 1 + x\,D^2.$$

(b) Prove, from part (a) or otherwise, that

$$d(n) = \frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 0.$$

(c) Prove that the number of lattice paths from $(0,0)$ to $(n,n)$, with steps $(0,1)$ or $(1,0)$, in which exactly one of each of the two types of steps lies below the line $y = x$, is given by

$$\frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 1.$$

(d) Prove that the number of lattice paths from $(0,0)$ to $(n,n)$, with steps $(0,1)$ or $(1,0)$, in which exactly $k$ of each of the two types of steps lies below the line $y = x$, is given by

$$\frac{1}{n+1}\binom{2n}{n},$$

for each $k = 0,\ldots,n$.

5. Let $b(m,n,k)$ be the number of $m \times n$ $\{0,1\}$–matrices in which no row or column consists entirely of 0's, and having exactly $k$ 1's among the entries of the matrix. Let

$$B(x,y,z) = \sum_{m,n,k\geq 0} b(m,n,k)\frac{x^m}{m!}\frac{y^n}{n!}z^k.$$

(a) Prove that

$$B(x,y,z) = e^{-(x+y)} \sum_{m,n\geq 0} \frac{x^m}{m!}\frac{y^n}{n!}(1+z)^{mn}.$$

(b) Deduce from part (a) that $B$ satisfies the partial differential equation

$$xy(1+\frac{\partial}{\partial x})(1+\frac{\partial}{\partial y})B = (1+z)\frac{\partial}{\partial z}B.$$

(c) Give a direct combinatorial proof of the result in part (b).