Enumeration Comprehensive Examination
Monday, June 24, 9 a.m. - Noon
2002

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1. Let $F(t,y,x) = \prod_{j\geq 0}(1-tyx^j)(1-tx^j)^{-1}$, and

$$\binom{m+k}{k}_x = \frac{\prod_{i=1}^{m+k}(1-x^i)}{\prod_{i=1}^{m}(1-x^i)\prod_{i=1}^{k}(1-x^i)}.$$

(a) Use Euler's device (a functional equation for F) to prove that

$$F(t,y,x) = 1 + \sum_{k\geq 1} t^k \prod_{i=1}^{k}(1-yx^{i-1})(1-x^i)^{-1}.$$

(b) Deduce from part (a) that

$$\prod_{j=0}^{m}(1-tx^j)^{-1} = 1 + \sum_{k\geq 1} t^k \binom{m+k}{k}_x.$$

(c) From part (b), give a combinatorial interpretation for

$$[x^n]\binom{m+k}{k}_x.$$

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2. Let I_n be the number of permutations of $\{1,\ldots,n\}$ that are involutions (i.e. permutations σ such that $\sigma(\sigma(i)) = i$, for $i = 1,\ldots,n$). Let $I(x) = \sum_{n\geq 0} I_n \frac{x^n}{n!}$.

(a) Prove that

$$I(x) = e^{x+\frac{x^2}{2}}.$$

(b) Deduce from part (a) that I_n satisfies the recurrence equation

$$I_n = I_{n-1} + (n-1)I_{n-2}, \quad n \geq 2$$

with initial conditions $I_0 = I_1 = 1$.

(c) Give a direct combinatorial argument to establish the recurrence in part (b).

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3. A *phylogenetic tree* (p–tree) on a set X is defined recursively as follows. If $X = \{v\}$ then v itself is the only p–tree on the set $\{v\}$. If T_1 and T_2 are p-trees on disjoint non–empty sets X_1 and X_2, respectively, then $\{T_1, T_2\}$ is a p–tree on the set $X_1 \cup X_2$. For example, $\{\{a,b\},c\}$, $\{\{a,c\},b\}$, $\{\{b,c\},a\}$ are the 3 p–trees on $\{a,b,c\}$, and $\{\{\{a,d\},c\},\{b,e\}\}$ is a p–tree on the set $\{a,b,c,d,e\}$. For each positive integer $n \geq 1$, determine the number of p-trees on the set $\{1,2,\ldots,n\}$.

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4. Consider lattice paths on integer points in two dimensions with two types of steps: *up* by two units, and *right* by one unit. Let g_n be the number of such paths from $(0,0)$ to $(2n,2n)$ which never go below the line $y = x$, $n \geq 0$. Let h_n be the number of such paths from $(0,0)$ to $(2n,2n)$ which never go below the line $y = x$, and which never touch the line $y = x$ except at $(0,0)$ and $(2n,2n)$, $n \geq 1$. Let $G = \sum_{n \geq 0} g_n x^n$ and $H = \sum_{n \geq 1} h_n x^n$.

(a) Prove that $G = (1-H)^{-1}$ and $H = xG^2$.

(b) From part (a), determine h_n, $n \geq 1$, and g_n, $n \geq 0$.

(c) Prove that the number of such paths from $(0,0)$ to $(2n,2n)$ which never go below the line $y = x$, and which touch the line $y = x$ exactly $m+1$ times (including at $(0,0)$ and $(2n,2n)$), is given by

$$\frac{m}{n}\binom{3n-m-1}{n-m}, \qquad n \geq m \geq 1.$$

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5. For a finite set V, let $V^* := \bigcup_{k=0}^{\infty} V^k$ denote the set of all finite sequences ("words") of elements of V. Consider a simple graph $G = (V,E)$. Define a relation $\sim$ on V^* as follows: for nonadjacent vertices $a, b \in V$ (so $\{a,b\} \notin E$) and words $\sigma, \rho \in V^*$, we let $\sigma a b \rho \sim \sigma b a \rho$. Let $\equiv$ be the reflexive and transitive closure of the relation $\sim$. For each $n \geq 0$, let $w_n(G)$ denote the number of equivalence classes of V^n under the relation $\equiv$. Finally, consider the ordinary generating function

$$W(G;\ x) := \sum_{n=0}^{\infty} w_n(G)x^n.$$

(a) For simple graphs G and H, $G \cup H$ is the graph with vertex–set $V(G) \cup V(H)$ and edge–set $E(G) \cup E(H)$. Prove that if G and H are vertex–disjoint, then

$$W(G \cup H;\ x) = W(G;\ x)W(H;\ x).$$

(b) For vertex–disjoint simple graphs G and H, the *join* of G and H is the graph $G \vee H$ with vertex–set $V(G \vee H) := V(G) \cup V(H)$ and edge–set

$$E(G \vee H) := E(G) \cup E(H) \cup \{\{g, h\} :\ g \in V(G) \text{ and } h \in V(H)\}.$$

Derive a formula for $W(G \vee H;\ x)$ in terms of $W(G;\ x)$ and $W(H;\ x)$.
