Combinatorial Enumeration
Comprehensive Examination

July 11, 2003

Three Hours: 1:00 pm – 4:00 pm

D.M. Jackson and D.G. Wagner

This examination consists of two parts, Part A and Part B.

Answer all five questions in Part A.

Select and solve one of the two questions in Part B.

Greater credit will be given for complete solutions than for fragmentary ones.

Part A

Answer all five questions in this part.

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1. Prove the following identities by enumerating certain sets of integer partitions, or otherwise.

(a)
$$\prod_{j=1}^{\infty} \frac{1}{1 - x^j y} = 1 + \sum_{k=1}^{\infty} \frac{x^{k^2} y^k}{\prod_{i=1}^{k}(1 - x^i)(1 - x^i y)}.$$

(b)
$$\left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{k^2}}{\prod_{i=1}^{k}(1 - x^{2i})}\right)^{-1} = 1 + \sum_{k=1}^{\infty} \frac{x^{k(k+1)/2}}{\prod_{i=1}^{k}(1 - x^i)}.$$

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2. For natural numbers n and k, let $S(n, k)$ denote the number of partitions of the set $\{1, 2, \ldots, n\}$ into k pairwise disjoint nonempty blocks, called a *Stirling number of the second kind*.

(a) Prove, bijectively or otherwise, that for $m, n \in \mathbb{N}$,

$$m^n = \sum_{k=0}^{n} k! S(n, k) \binom{m}{k}.$$

(b) For a sequence $a_0, a_1, a_2, \ldots$ of real numbers, define another sequence $b_0, b_1, b_2, \ldots$ as follows: for all $j \in \mathbb{N}$,

$$b_j := \sum_{i=0}^{j} \binom{j}{i} a_i.$$

Prove that for all $i \in \mathbb{N}$,

$$a_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} b_j.$$

(c) Deduce a formula for $S(n, k)$ from parts (a) and (b).

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3. For $m, n, k \in \mathbb{N}$, let $a_{m,n,k}$ denote the number of m–by–n matrices with entries in $\{0, 1\}$, with no rows or columns consisting entirely of 0s, and with exactly k 1s.

(a) Give an expression for the generating series

$$A(x, y, z) := \sum_{m,n,k \geq 0} a_{m,n,k} \frac{x^m}{m!} \frac{y^n}{n!} z^k.$$

(b) By considering $F(x, y, z) := \exp(x + y)A(x, y, z)$ or otherwise, deduce a partial differential equation satisfied by $A(x, y, z)$, and show that $\{a_{m,n,k} : m, n, k \geq 0\}$ satisfies a linear recurrence relation.

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4. Let $t, x_1, x_2, x_3, \ldots$ be independent indeterminates, and for $k \in \mathbb{N}$ let

$$e_k := [t^k] \prod_{i=1}^{\infty} (1 + tx_i)$$

(that is, e_k is the k–th elementary symmetric function of the $\{x_i\}$). Define a homomorphism

$$\Delta : \mathbb{Q}[e_1, e_2, \ldots] \longrightarrow \mathbb{Q}[[z]]$$

by setting $\Delta(e_k) := z^k/k!$ and extending the definition linearly and multiplicatively. Prove that for any symmetric function $F(e_1, e_2, \ldots) \in \mathbb{Q}[e_1, e_2, \ldots]$,

$$[x_1 x_2 \ldots x_n] F(e_1, e_2, \ldots) = \left[\frac{z^n}{n!}\right] \Delta F(e_1, e_2, \ldots).$$

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5. Let $c_k(T, v)$ denote the number of nodes with exactly $k \in \mathbb{N}$ children in a rooted labelled tree (T, v). Let $\bar{c}_k(n)$ denote the average value of $c_k(T, v)$ among all n^{n-1} rooted labelled trees on the set $\{1, 2, \ldots, n\}$. Prove that for $0 \leq k \leq n$:

$$\bar{c}_k(n) = \frac{n^2}{(n-1)^{k+1}} \binom{n-1}{k} \left(1 - \frac{1}{n}\right)^n.$$

Part B

Select and solve one of the two questions in this part.

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6. Let p be a prime number, c a positive integer, and $q := p^c$. For $k \in \mathbb{N}$ let

$$[k]_q := 1 + q + q^2 + \cdots q^{k-1}$$

and

$$[k]!_q := [k]_q [k-1]_q \cdots [3]_q [2]_q [1]_q.$$

(a) Show that the number of ordered bases of an m–dimensional vector space over the finite field $GF(q)$ is

$$q^{m(m-1)/2} (q-1)^m [m]!_q.$$

(b) Show that the number of k–dimensional subspaces of $GF(q)^n$ is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q [n-k]!_q}.$$

(c) By enumerating linear transformations from $GF(q)^n$ to a vector space over $GF(q)$ of **cardinality** x, or otherwise, show that

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=1}^{n-k} (x - q^{i-1}).$$

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7. In a sequence $\sigma = i_1 i_2 \cdots i_k$ of positive integers, $i_j i_{j+1}$ is called a *rise* if $i_j < i_{j+1}$, and a *non–rise* if $i_j \geq i_{j+1}$. The *pattern of* σ is the monomial in $\mathbb{Q}\langle\langle u, d \rangle\rangle$ that specifies the sequence of rises (marked by u) and non–rises (marked by d) in σ. For example, the pattern of 2 3 3 2 is udd.

Let $\varphi : \mathbb{Q}\langle\langle u, d \rangle\rangle \longrightarrow \mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ be the mapping defined as follows. Let $a, b \in \mathbb{Q}\langle\langle u, d \rangle\rangle$, let $\lambda, \mu \in \mathbb{Q}$, and let $w := u + d$. Then
(i) $\varphi(\lambda a + \mu b) = \lambda\varphi(a) + \mu\varphi(b)$;
(ii) $\varphi(awb) = \varphi(a)\varphi(b)$;
(iii) $\varphi(u^{k-1}) = e_k(x_1, x_2, \ldots)$, in which $e_k := [t^k] \prod_{i=1}^{\infty}(1 + tx_i)$ is the k–th elementary symmetric function of the $\{x_i\}$.

(a) For any pattern p, show that $\varphi(p)$ is the ordinary generating series for all sequences with the pattern p, in which the indeterminate x_i marks occurrences of the positive integer i. (So, for example, the sequence 2 3 3 2 contributes $x_2^2 x_3^2$ to $\varphi(udd)$.)

(b) By finding $q, a, b \in \mathbb{Q}\langle\langle u, d \rangle\rangle$ such that $1 - ud = q + awb$, show how to use the properties of φ to prove that

$$\varphi((1 - ud)^{-1}) = \frac{\sum_{k=0}^{\infty}(-1)^k e_{2k+1}}{\sum_{k=0}^{\infty}(-1)^k e_{2k}}.$$

(c) (For this part you may use the statement of Question 4, whether or not you have proved it.) Deduce, with the aid of the result of Question 4, or otherwise, that the number of permutations of $\{1, 2, \ldots, 2n + 1\}$ with the pattern $(ud)^n$ is

$$\left[\frac{z^{2n+1}}{(2n + 1)!}\right] \tan(z).$$

Check this answer explicitly for $n = 1$ (for permutations of $\{1, 2, 3\}$).