Enumeration Comprehensive

Friday 10th June 2005

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Provide solutions to at least five of the seven questions.

Greater credit will be given to well–reasoned and complete solutions than to fragments or sketches of solutions.

1. A *trivalent labelled tree* is a tree with vertex–set $\{1, \ldots, n\}$ for some positive integer n in which each vertex has degree either 1 or 3. Show that if $n = 2k$ then the number of trivalent labelled trees with n vertices is
$$\frac{(2k-2)!}{2^{k-1}}\binom{2k}{k-1}.$$

2. (a) Fix an integer $b \geq 2$. Let $\mathcal{A}$ be the set of integer partitions in which each part occurs at most $b-1$ times. Let $\mathcal{B}$ be the set of integer partitions in which no part is divisible by b. Show that for every integer n, the number of partitions of n in $\mathcal{A}$ equals the number of partitions of n in $\mathcal{B}$.

   (b) In the special case $b = 2$, describe explicitly a weight–preserving bijection between the sets $\mathcal{A}$ and $\mathcal{B}$ in part (a). (A proof of correctness is not required.)

   (c) Provide a bijection as in part (b) for the general case $b \geq 2$.

3. Let $\gamma_k = [t^k]\prod_{i\geq 1}(1-tx_i)^{-1}$ where $t, x_1, x_2, \ldots$ are indeterminates.

   (a) Let $\phi(\gamma_1, \gamma_2, \ldots)$ be a formal power series in $\gamma_1, \gamma_2, \ldots$. Prove that
$$[x_1 \cdots x_n]\phi(\gamma_1, \gamma_2, \ldots) = \left[\frac{x^n}{n!}\right]\phi\left(\frac{x}{1!}, \frac{x^2}{2!}, \ldots\right).$$

   (b) Prove that the ordinary generating series for the number of alternating sequences of even length is
$$\left(\sum_{k\geq 0}(-1)^k\gamma_{2k}\right)^{-1},$$
   and thence find the generating series for the number of alternating permutations of even length.

4. A *proper 2-cover of order* k of $\{1, \ldots, n\}$ is a set $\mathcal{B}$ of non-empty and pairwise mutually distinct subsets $\mathcal{B}_1, \ldots, \mathcal{B}_k$ of $\{1, \ldots, n\}$ such that each element of $\{1, \ldots, n\}$ appears in exactly two members of $\mathcal{B}$. Prove that the number $a(k, n)$ of such covers is given by

$$\left[\frac{x^k}{k!}\frac{y^n}{n!}\right] A(x, y)$$

where

$$A(x, y) = e^{-x^2(e^y-1)/2-x} \sum_{k,n\geq 0} \binom{k}{2}^n \frac{x^k}{k!}\frac{y^n}{n!}.$$

5. (a) Let α and x be indeterminates. Find a formal power series $f(y)$ such that $e^{\alpha x} = f(xe^{-x})$.

(b) Let β be another indeterminate. From part (a) or otherwise, prove that

$$(\alpha + \beta)(n + \alpha + \beta)^{n-1} = \alpha\beta \sum_{k=0}^{n} \binom{n}{k} (k + \alpha)^{k-1}(n - k + \beta)^{n-k-1}.$$

6. (a) An *inversion* of a permutation $a_1 a_2 \ldots a_n$ is a pair of indices (i, j) such that $1 \leq i < j \leq n$ and $a_i > a_j$. Let $\mathrm{inv}(\sigma)$ denote the number of inversions of the permutation $\sigma \in \mathfrak{S}_n$, and for an indeterminate q define the polynomial

$$[n]!_q := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)}.$$

Give a combinatorial proof that for $n \geq 1$,

$$[n]!_q = [n-1]!_q(1 + q + q^2 + \cdots + q^{n-1}).$$

(b) Let $q = p^c$ be a prime power. Let V be an n–dimensional vector space over the finite field $\mathrm{GF}(q)$. Show that the number of ordered bases of V is

$$(q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1}).$$

(c) Show that the number of k–dimensional subspaces of V is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q [n-k]!_q},$$

with the notation defined in part (a).

7. (a) Let $Z_{\mathfrak{S}_n}(x_1, x_2, \ldots, x_n)$ denote the cycle index polynomial of the symmetric group $\mathfrak{S}_n$, for each $n \geq 0$. Give a formula for the generating series

$$F(t;\ x_1, x_2, \ldots) = \sum_{n=0}^{\infty} t^n Z_{\mathfrak{S}_n}(x_1, x_2, \ldots, x_n).$$

(b) Let c_n denote the number of rooted (but unlabelled) trees on n vertices. Using part (a) or otherwise, find an expression giving each c_n in terms of $c_0, c_1, c_2, \ldots, c_{n-1}$.