COMPREHENSIVE EXAM: ENUMERATION

1:00–4:00 pm, June 14, 2006

1(a) Let $p_m(n)$ be the number of partitions of n in which the number of parts is at most m, for $n, m \geq 0$. Prove that

$$\sum_{n\geq 0} p_m(n)x^n = \prod_{i=1}^{m} \frac{1}{1-x^i}.$$

(b) Let $s(n)$ be the number of self-conjugate partitions of n, for $n \geq 0$. Prove that

$$\sum_{n\geq 0} s(n)x^n = \prod_{j\geq 0}(1+x^{2j+1}).$$

(c) Let $s_m(n)$ be the number of self-conjugate partitions of n which the number of parts is at most m, for $n, m \geq 0$. Find a formula for $\sum_{n\geq 0} s_m(n)x^n$.

2. Fix a positive integer k. For a permutation σ, let $c(\sigma, k)$ be the number of cycles in σ of length exactly k.

(a) Obtain an algebraic formula for the bivariate exponential generating function

$$S(x,y) = \sum_{n=0}^{\infty} \left(\sum_{\sigma\in \mathcal{S}_n} y^{c(\sigma,k)} \right) \frac{x^n}{n!}.$$

(b) Show that the average number of cycles of length k among all $n!$ permutations in $\mathcal{S}_n$ is

$$\begin{cases} 1/k & \text{if } k \leq n, \\ 0 & \text{if} k > n. \end{cases}$$

3(a) Let α and x be indeterminates. Find a formal power series f such that $f(xe^{-x}) = e^{\alpha x}$.

(b) Let β be another indeterminate. From part (a) or otherwise, prove that, for $n \geq 0$,

$$(\alpha+\beta)(n+\alpha+\beta)^{n-1} = \alpha\beta \sum_{k=0}^{n} \binom{n}{k} (k+\alpha)^{k-1}(n-k+\beta)^{n-k-1}.$$

4. A *cactus* is a connected graph such that each edge is in at most one cycle. Equivalently, it is a connected graph in which each block (2–connected component) is either an edge or a cycle. An *oriented cactus* is a cactus in which each cycle has been directed in one of its two strongly connected orientations.

(a) Show that for all $n \geq 1$, the number of oriented cacti on the vertex-set $\{1, \ldots, n\}$ is

$$(n-1)! \sum_{k=0}^{n-1} \frac{n^{k-1}}{k!} \binom{n-2}{n-1-k}.$$

(b) Derive a functional equation that implicitly determines the exponential generating function for the class of rooted unoriented cacti.

5(a) A permutation $a_1\ a_2\ \ldots\ a_n$ of the set $\{1, \ldots, n\}$ is said to be *123–avoiding* if there do not exist three indices $1 \leq i < j < k \leq n$ such that $a_i < a_j < a_k$. Prove that the number of 123–avoiding permutations of length n is $\frac{1}{n+1}\binom{2n}{n}$.

(b) A permutation $a_1\ a_2\ \ldots\ a_n$ of the set $\{1, \ldots, n\}$ is said to be *231–avoiding* if there do not exist three indices $1 \leq i < j < k \leq n$ such that $a_k < a_i < a_j$. Prove that the number of 231–avoiding permutations of length n is $\frac{1}{n+1}\binom{2n}{n}$.