COMPREHENSIVE EXAM – ENUMERATION
July 7, 2008

(Total marks = 100)

1(a) Let $a_{n,k}$ denote the number of compositions of n in which k of the parts are equal to 1, for $n \geq k \geq 0$. Prove that [6]

$$\sum_{n \geq k \geq 0} a_{n,k} u^k x^n = \frac{1-x}{1-2x-(u-1)x(1-x)}.$$

(b) From (a), determine μ_n, the average number of parts equal to 1 in compositions of n, for $n \geq 0$. [5]

(c) Let b_n denote the number of compositions of n in which there are no consecutive 1's among the parts. Prove that [6]

$$\sum_{n \geq 0} b_n x^n = \frac{1-x^2}{1-x-x^2-x^3}.$$

2(a) Determine the number of rooted, labelled trees on vertex-set $\{1, \ldots, n\}$, in which the root vertex has degree k. [9]

(b) Determine the number of rooted, labelled trees on vertex-set $\{1, \ldots, n\}$, in which vertex n is a leaf, and is not the root. [7]

(c) Determine the number of rooted, labelled trees on vertex-set $\{1, \ldots, n\}$, in which the root vertex is larger (i.e. has larger label) than all of its neighbours. [10]

3(a) Let c_n denote the number of lattice paths (with steps $(0,1)$ and $(1,0)$) from $(0,0)$ to (n,n), which never descend below the line $y = x$, $n \geq 0$. Define $C(x) = \sum_{n \geq 0} c_n x^n$. Prove that [7]

$$C(x) = 1 + xC(x)^2,$$

and hence determine an explicit formula for c_n, $n \geq 0$.

(b) Prove that the generating series with respect to number of upsteps, for lattice paths from $(0,0)$ to a point on the line $y = x + k$, which never descend below the line $y = x$, is given by $x^k C(x)^{k+1}$. From this generating series, give an explicit expression for the number of lattice paths from $(0,0)$ to $(n, n+k)$, which never descend below the line $y = x$, $n \geq 0$. [8]

(c) Let $d_{n,k}$ be the number of lattice paths from $(0,0)$ to (n,n) with k upsteps below the line $y=x$ and $n-k$ upsteps above the line $y=x$. Prove that [10]

$$\sum_{n\geq k\geq 0} d_{n,k}x^{n-k}y^k = \frac{1}{1-xC(x)-yC(y)}.$$

Deduce from this generating series that $d_{n,k}$ is independent of k (and hence that $d_{n,k} = d_{n,0} = c_n$ for all k).

4(a) Prove the identity [7]

$$\prod_{k=0}^{d}(1-tq^k)^{-1} = \sum_{n=0}^{\infty}\binom{n+d}{d}_q t^n,$$

where $\binom{n+d}{d}_q = \prod_{j=1}^{d}\frac{1-q^{n+j}}{1-q^j}$.

(b) Hence or otherwise, prove that $[q^k]\binom{n+d}{d}_q$ is the number of partitions of k with at most d parts and with largest part at most n. [5]

(c) Define a *composition of n into partitions* to be a k-tuple $(\lambda_1,\ldots,\lambda_k)$ for any $k\geq 0$, such that $\lambda_i = \lambda_{i1}\geq\cdots\geq\lambda_{id_i} > 0$ is a partition of a positive integer, and $|\lambda_1|+\cdots+|\lambda_k| = n$. The partitions λ_i are called the *components* of the composition into partitions. (Eg. $(2\,,4{\geq}2{\geq}1)$ and $(1{\geq}1{\geq}1{\geq}1\,,1\,,1\,,1{\geq}1\,,1)$ are compositions of 9 into partitions, with two components and five components respectively.) [10]

Let a_n be the number of compositions of n into partitions in which an even number of components have an even number of parts.

Let b_n be the number of compositions of n into partitions in which an odd number of components have an even number of parts.

Let c_n be the number of partitions of n with distinct parts.

Prove that $a_n = b_n + c_n$ for each $n\geq 1$.

5. Let c_n be the number of $2\times n$ matrices A (with (i,j)-entry $A_{i,j}$) having the following properties: [10]

- Each number from $\{1,\ldots,n\}$ appears twice in A.
- $A_{1,j}\leq A_{2,j}$ for all j.
- $A_{i,j}\leq A_{i,j+1}$ for $i=1,2$, $j=1,\ldots,n-1$.

Prove that

$$\sum_{n\geq 0} c_n x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}.$$