COMPREHENSIVE EXAMINATION – ENUMERATION

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Monday, June 22, 2009, 2:00-5:00.

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1. [10 pts.] Let g_n be the number of plane planted trees with n nodes, in which every node has an even number of children. Prove that

$$g_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{2k+1}\binom{3k}{k} & \text{if } n = 2k+1. \end{cases}$$

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2. For natural numbers n and k, let $S(n,k)$ denote the number of partitions of the set $\{1, 2, \ldots, n\}$ into k pairwise disjoint nonempty blocks, called a *Stirling number of the second kind.*

(a) [8 pts.] Prove, bijectively or otherwise, that for $m, n \in \mathbb{N}$,

$$m^n = \sum_{k=0}^{n} k! S(n,k) \binom{m}{k}.$$

(b) [5 pts.] For a sequence $a_0, a_1, a_2, \ldots$ of real numbers, define another sequence $b_0, b_1, b_2, \ldots$ as follows: for all $j \in \mathbb{N}$,

$$b_j = \sum_{i=0}^{j} \binom{j}{i} a_i.$$

Prove that for all $i \in \mathbb{N}$,

$$a_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} b_j.$$

(c) [4 pts.] Deduce a formula for $S(n,k)$ from parts (a) and (b).

3(a) [8 pts.] Let α and x be indeterminates. Find a formal power series f such that $f(xe^{-x}) = e^{\alpha x}$.

(b) [11 pts.] Let β be another indeterminate. From part (a) or otherwise, prove that, for $n \in \mathbb{N}$,

$$(\alpha+\beta)(\alpha+\beta+n)^{n-1} = \alpha\beta \sum_{k=0}^{n} \binom{n}{k} (\alpha+k)^{k-1}(\beta+n-k)^{n-k-1}.$$

4(a) [8 pts.] Let σ be a permutation of the set $\{1, 2, \ldots, n\}$ chosen uniformly at random, and let $1 \leq k \leq n$. What is the probability that the cycle of σ that contains 1 has length exactly k?

(b) [11 pts.] A *phylogeny of order* n is a tree in which the leaves are labelled by the set $\{1, 2, \ldots, n\}$ and each non-leaf vertex has degree three. Let a_n denote the number of (isomorphism classes of) phylogenies of order n. Thus, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, and $a_4 = 3$. Show that for all $n \geq 3$,

$$a_n = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-5),$$

the product of the first $n-2$ positive odd integers.

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5. [15 pts.] An *endofunction* on a set X is any function $f : X \to X$. An element $v \in X$ is *recurrent* for $f : X \to X$ if there is a positive integer $m \geq 1$ for which $f^{[m]}(v) = v$. (Here, $f^{[m]}$ denotes the m-fold composition of f with itself.) For each $1 \leq k \leq n$, derive a formula for the number of endofunctions on an n-element set with exactly k recurrent vertices, and conclude that

$$\sum_{k=1}^{n} \frac{(n-1)!k}{(n-k)!n^k} = 1.$$

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6. Consider the formal power series

$$F_m(q, t) = \prod_{j=0}^{m} \frac{1}{1 - q^j t} = \sum_{n=0}^{\infty} c_{m,n}(q) t^n.$$

(a) [8 pts.] State and prove a combinatorial interpretation of $F_m(q, t)$ as an ordinary generating series. State the combinatorial interpretation of $c_{m,n}(q)$.

(b) [12 pts.] By comparing $F_{m+1}(q, t)$ and $F_m(q, qt)$, or otherwise, obtain a linear recurrence relation for $c_{m,n}(q)$. Solve this recurrence to obtain an explicit formula for $c_{m,n}(q)$ as a polynomial in q.
