

# Enumeration Comprehensive Examination

Friday, June 1st 2012, 9:00 am–12:00 noon

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1. (a) Let  $S(n)$  be the number of partitions of  $n$  with distinct parts. Prove that [5]

$$\sum_{n \geq 0} S(n)x^n = \prod_{i \geq 1} (1 + x^i).$$

- (b) Let  $Q(n)$  be the number of partitions of  $n$  with only odd parts. Prove that [7]

$$\sum_{n \geq 0} Q(n)x^n = \prod_{i \geq 1} (1 + x^i).$$

- (c) For  $k, \ell \geq 2$ ,  $n \geq 0$ , let  $P(n; k, \ell)$  be the number of partitions of  $n$  in which no part is divisible by  $k$ , and each part occurs with multiplicity at most  $\ell - 1$ . For example,  $P(n; 2, 2)$  is the number of partitions of  $n$  with distinct, odd parts. Prove that  $P(n; k, \ell) = P(n; \ell, k)$ . [8]

2. Let  $m, n \in \mathbb{N}$ . Prove that the number of surjective functions from  $\{1, \dots, m\}$  to  $\{1, \dots, n\}$  is equal to [12]

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k^m.$$

3. A *plane trivalent tree* is a planar embedding of a tree with unlabelled vertices, in which every vertex has degree 1 or 3. A *rooted plane trivalent tree* is a plane trivalent tree with one special edge that is oriented; this edge is called the *root* and we depict it with an arrow. We will consider two plane trivalent trees (rooted or otherwise) to be equivalent if they are isotopic, i.e. if it is possible to continuously deform one into the other via planar embeddings of underlying graph. For example, the two rooted plane trivalent trees shown below are *not* equivalent; however if we forget which edge is the root, they become equivalent unrooted trees.



- (a) Let  $r_n$  denote the number of rooted plane trivalent trees with  $n$  vertices. Let  $R(x) = \sum_{n \geq 0} r_n x^n$ . Show that [10]

$$R(x) = x^2(R(x) + 1)^2,$$

and hence prove that  $r_{2n} = \frac{1}{n+1} \binom{2n}{n}$ ,  $n \geq 1$ .

- (b) Let  $t_n$  denote the number of (unrooted) plane trivalent trees with  $n$  vertices. Prove that [12]

$$\sum_{k \geq 0} t_k x^k = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} \left( \frac{x^{2n+2}}{n+2} + \frac{x^{4n+2}}{2} + \frac{2x^{6n+4}}{3} \right).$$

4. (a) Let  $C(m, n; k)$  denote the number of cycles of length  $k$  in the complete bipartite graph  $K_{m,n}$ . Obtain a closed formula for the generating series [12]

$$\sum_{m,n,k \geq 0} C(m, n; k) \frac{x^m y^n z^k}{m! n!}.$$

- (b) Let  $\mathcal{G}_{m,n}$  be the set of all subgraphs of  $K_{m,n}$ . If a graph  $G$  is chosen uniformly at random from  $\mathcal{G}_{m,n}$ , prove that the expected number of cycles in  $G$  is [8]

$$\left[ \frac{x^m y^n}{m! n!} \right] e^{x+y} \left( \frac{1}{2} \log(1 - \frac{xy}{4})^{-1} - \frac{xy}{8} \right).$$

5. Let  $q$  be a prime power, and  $n \in \mathbb{N}$ . For  $k \geq 0$ , define

$$[k]!_q = \prod_{i=1}^k (1 + q + \cdots + q^{i-1}).$$

Let  $V$  be an  $n$ -dimensional vector space over  $\mathbb{F}_q$ , the field with  $q$ -elements.

- (a) For  $0 \leq k \leq n$ , prove that the number of  $k$ -dimensional linear subspaces of  $V$  is equal to [9]

$$\binom{n}{k}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q}.$$

- (b) A *full flag* in  $V$  is an  $(n+1)$ -tuple  $(F_0, F_1, \dots, F_n)$  of linear subspaces of  $V$ , such that [7]

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n$$

and  $\dim(F_i) = i$  for  $i = 0, \dots, n$ . (Hence  $F_0$  is the 0-subspace, and  $F_n = V$ .) Prove that the number of full flags in  $V$  is equal to  $[n]!_q$ .

- (c) Let  $\mathcal{S}_n$  denote the set of permutations of  $\{1, \dots, n\}$ . An *inversion* of a permutation  $\sigma \in \mathcal{S}_n$  is a pair  $(i, j)$  such that  $1 \leq i < j \leq n$  and  $\sigma(i) > \sigma(j)$ . Let  $\text{inv}(\sigma)$  denote the number of inversions of  $\sigma$ . [10]

Let  $\mathcal{A}_n \subset \text{GL}_n(\mathbb{F}_q)$  be the set of invertible matrices over  $\mathbb{F}_q$  with the following two properties:

- In each row the rightmost non-zero entry is equal to 1. The rightmost non-zero entry in any row is called a *pivot*.
- Every entry that is below a pivot is also equal to 0. (In particular, there cannot be two pivots in the same column.)

Establish a bijection between  $\mathcal{A}_n$  and the set of full flags in  $V$ . Hence, or otherwise, prove that

$$[n]!_q = \sum_{\sigma \in \mathcal{S}_n} q^{\text{inv}(\sigma)}.$$