UNIVERSITY OF WATERLOO | FACULTY OF MATHEMATICS
Department of Combinatorics and Optimization

C&O Comprehensive Exam Syllabus for Combinatorial Enumeration

The body of knowledge which makes up enumeration is built on certain foundational examples. To reflect this, the topics which can appear on the exam are divided into **tools**, the theorems, concepts and techniques students should know how to use; objects, the main combinatorial **objects** each with its own bijections, definitions, and standard questions; and **problem types**, the different kinds of questions students should know how to answer.

There are two primary references which define the scope of the material that students are required to know, and four alternate references, each with different strengths that students may find helpful for additional examples and exercises and for their different perspectives. The alternate references are not required reading; students may use them as they see fit.

References:

Primary References: The vast majority of syllabus material below can be found in these references. For topics where the treatment in these sources may be insufficient, an *additional reference* is listed.

[W] D. Wagner. *CO 330 Course notes*, Chapters 3-12.

[S] Stanley, *Enumerative Combinatorics, Volume 1*, Wadsworth 1986 (reprinted Cambridge 1997), Sections 1.1-1.3, 2.1-2.4, 2.6, 3.7-3.8, 4.7.

Alternate References: Students may find these references to be a useful supplement or alternative on some topics.

[A] M. Aigner, *A Course in Enumeration*, GTM 238, Springer 2007, Chapters 1,2,3,5.

[BLL] Bergeron, Labelle, and Leroux, *Combinatorial Species and Tree-like Structures*, Cambridge 1998. Sections 1.1-1.4, 2.1, 2.3, 2.4, 3.1, 5.1.

[FS] Flajolet and Sedgewick, *Analytic Combinatorics*, Cambridge 2009, Chapters I, II, III, V.6.

[GJ] Goulden and Jackson, *Combinatorial Enumeration*, Wiley 1983 (reprinted Dover 2004), Sections 1.1, 1.2.1-1.2.7, 2.1-2.7, 3.1-3.4.

Topics:

1. Students should be comfortable with the following tools.
   - Ordinary generating functions
   - Exponential generating functions
   - Basic language of combinatorial species
   - Multivariate and mixed generating functions (additional reference: [GJ, 3.4] or [BLL, 2.4])
   - Inclusion-exclusion (additional reference: [GJ, 2.2.28-2.2.30])
   - Sign reversing involutions
   - Möbius inversion

- Lagrange implicit function theorem
- Transfer matrix method
- Rook polynomials

2. Students should be familiar with the following core combinatorial objects, knowing the definitions and key results in each case. Other objects may appear on exams, but in that case definitions would be given and students would not be expected to have prior experience with these objects.
   - Partitions
   - Compositions
   - Trees, including rooted trees, labelled trees, planted trees
   - Permutations
   - Lattice paths
   - Strings
   - Set partitions
   - Functions, including endofunctions, injective functions, surjective functions
   - Graphs
3. Students should additionally be comfortable with the following key problem types.
   - Enumeration of combinatorial objects by generating functions
   - Bijections between classes of combinatorial objects
   - Proving identities by bijections and combinatorial interpretations
   - Counting with q-binomial coefficients and other q-analogues
   - Enumeration involving multivariate generating functions, with any mix of ordinary and exponential flavours
   - Calculating averages by bivariate generating functions
   - Finding recursive structures and building combinatorial objects from recursive structures
   - Theory of formal power series
   - Enumeration of vector spaces over finite fields