First-Stage PhD Comprehensive Examination

in

CONTINUOUS OPTIMIZATION

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MC 4044, Wednesday, May 31, 2017, 1pm – 4pm **(3 hours)**

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1. Consider linear programming problems in the form

$$\text{(LP)} \qquad \begin{array}{rl} p^* := \min & c^\top x \\ \text{s.t.} & Ax = b \\ & x \in \mathbb{R}^n_+, \end{array}$$

where A is a full row rank m-by-n matrix. In your answers, you may use a Strong Duality Theorem for LP, without proof, provided you state it clearly and correctly.

   (a) Derive the dual of LP.

   (b) Show that the feasible set of LP is a polyhedron and define an "extreme point" of a polyhedron.

   (c) Give an example of an LP problem in the dual form of LP that is feasible but does not have an extreme point.

   (d) Suppose that p^* is finite and that there is a strictly feasible solution ($\exists \bar{x} \in \mathbb{R}^n_{++}$ such that $A\bar{x} = b$). Show that the optimal solution set of the dual is a nonempty, compact, convex set.

   (e) Suppose that p^* is finite. Is it possible to find an instance of the above family of LPs where p^* is finite and both the feasible solution set of LP and of its dual are bounded sets? Prove all your claims.

2. (a) Let $f : \mathbb{R}^n \to (-\infty, +\infty]$. Define the Fenchel-Legendre conjugate f^* of f.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be the exponential function, i.e., $f(x) := \exp(x)$. Compute the Fenchel-Legendre conjugate of f.

   (c) Define what it means for $f : \mathbb{R}^n \to (-\infty, +\infty]$ to be a *proper, closed convex function*.

   (d) Let $f : \mathbb{R}^n \to (-\infty + \infty]$ be a proper, closed convex function. Then, prove that f has bounded level sets iff f^* is continuous at 0.

3. Let $q : \mathbb{R}^n \to \mathbb{R}$,

$$q(x) := \frac{1}{2}x^\top Qx - b^\top x,$$

   be a quadratic function with Q, b appropriate sized given matrices.

   (a) Let

$$p^* := \min_{x \in \mathbb{R}^n} q(x).$$

   Provide a characterization (with a proof) for $p^* > -\infty$, i.e., for p^* being finite valued.

   (b) Consider the quadratic programming problem (QP)

$$\begin{array}{rl} \min & q(x) \\ \text{s.t.} & Ax = c \\ & x \in \mathbb{R}^n\,, \end{array}$$

   where $c \in \mathbb{R}^m$, $m < n$, and A is an appropriate matrix. Show that x^* is a local minimizer of QP if, and only if, x^* is a global minimizer of QP.[1]

   (c) Consider the trust region subproblem

$$\text{(TRS)} \qquad \begin{array}{rl} p^* := \min & q(x) \\ \text{s.t.} & \|x\|_2 \leq \delta \\ & x \in \mathbb{R}^n\,, \end{array}$$

   where $\delta > 0$.

   i. Show that p^* is attained.

   ii. Provide a characterization (with a proof) for x^* being a global minimizer of TRS.

   iii. When is the global minimizer of TRS unique? Why?

[1] No convexity is assumed.

4. (a) Let $f : \mathbb{R}^n \to (-\infty, +\infty]$. Define the notion of f being a *coercive* function. Then let f be a coercive function and prove that every level set of f is bounded.

   (b) Suppose the $n := 1$ and f is differentiable and suppose that

$$\lim_{|x|\to\infty} \frac{f(x)}{|x|} = +\infty.$$

   Show that

$$\{f'(x) : x \in \mathbb{R}\} = \mathbb{R}.$$

   (c) Let $f : \mathbb{R}^n \to (-\infty, +\infty]$, and $\bar{x} \in \mathbb{R}^n$. Define the subdifferential $\partial f(\bar{x})$.

   (d) Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be given by $f(x) := \|x\|_\infty$. Compute $\partial f(0)$. Prove all your claims.

5. Consider the abstract convex programming problem

$$\text{(CP)} \qquad \begin{array}{rl} p^* := \min & f(x) \\ \text{s.t.} & g(x) \preceq_K 0 \\ & x \in \Omega, \end{array}$$

   where f is convex on the convex set Ω, and $g : \Omega \to \mathbb{R}^m$ is K-convex with respect to the closed convex cone K.

   (a) State the Slater constraint qualification for CP.

   (b) Suppose that p^* is finite and the Slater condition holds. State the Lagrange multiplier necessary and sufficiency theorem.[2]

   (c) Suppose that there are Lagrange multiplier vectors λ_0, λ_1 for right-hand side perturbations of CP z_0, z_1, respectively, with optimal solutions x_0, x_1, respectively, i.e., for the constraints $g(x) \preceq_K z_i$, $i \in \{1, 2\}$, respectively. State and prove a sensitivity analysis theorem, i.e., one that provides bounds for $f(x_0) - f(x_1)$.

[2] There should be separate statements for with and without attainment and for necessity and sufficiency.