Enumeration Comprehensive Examination

Wednesday, June 1st 2016, 1:30pm – 4:30pm

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Instructions. Attempt to answer all questions. Greater credit will be given to complete, well-reasoned solutions than to fragmentary or partial solutions. If question X comes *before* question Y in the exam, then you may (for full credit) use the result of X to in your solution to Y even if you did not solve X; however, if you use the result of Y in your solution to X, credit will only be given if Y is solved correctly.

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1. [8/8 pts.] Define, with justification, a natural class of structures (a.k.a. species)

(a) ... $\mathcal{Q}$ with exponential generating function $Q(x) = \big(1/(1-x)\big)^{1/(1-x)}$.

(b) ... $\mathcal{R}$ with exponential generating function $R(x)$ satisfying $R(x) = x\cosh(R(x))$. (Recall: $\cosh x = (e^x + e^{-x})/2$.)

2. [8/10 pts.] For $k, n \geq 0$, let $d_{n,k}$ denote the number of lattice paths from $(0,0)$ to (n,n) that have at least one point on the line $y = x - k$ but no points strictly below it. (If $k = 0$, these are Dyck paths.)

(a) Let $C(x) = \sum_{n\geq 0} d_{n,0} x^n$. Prove that

$$C(x) = 1 + xC(x)^2\,,$$

and hence find an explicit formula for $d_{n,0}$.

(b) Let $D(x,y) = \sum_{n,k\geq 0} d_{n,k} x^n y^k$. Prove that

$$D(x,y) = \frac{C(x)}{1 - xyC(x)^2}\,,$$

and hence find an explicit formula for $d_{n,k}$.

3. [9/9 pts.] Let m be a positive integer. For an m-ary string $\alpha \in [m]^*$ and $i \in [m]$, let $w_i(\alpha)$ denote the number of i's in α. Write $\mathbf{x}^{\mathbf{w}(\alpha)} = x_1^{w_1(\alpha)} x_2^{w_2(\alpha)} \cdots x_m^{w_m(\alpha)}$.

Let $\Sigma \subset [m]^2$ be a set of m-ary strings of length 2, and let $\overline{\Sigma} = [m]^2 \setminus \Sigma$ be the complementary set. Let $\mathcal{G}$ (respectively $\overline{\mathcal{G}}$) be the set of all m-ary strings with the property that every substring of length 2 belongs to Σ (respectively $\overline{\Sigma}$). (Note that $\mathcal{G}$ and $\overline{\mathcal{G}}$ both include all m-ary strings of length 0 or 1.) Consider the two generating series $G(\mathbf{x}) = \sum_{\alpha\in\mathcal{G}} \mathbf{x}^{\mathbf{w}(\alpha)}$, $\overline{G}(\mathbf{x}) = \sum_{\alpha\in\overline{\mathcal{G}}} \mathbf{x}^{\mathbf{w}(\alpha)}$.

(a) Compute $G(\mathbf{x})$ and $\overline{G}(\mathbf{x})$ in the case where $\Sigma = \{11, 22, \ldots, mm\}$.

(b) Prove that $\overline{G}(x_1, \ldots, x_m) = G(-x_1, \ldots, -x_m)^{-1}$.

4. [10/8 pts.] For a permutation $\sigma \in S_n$ and $i \in [n-1]$, we say that i is a *descent* of σ if $\sigma(i) > \sigma(i+1)$. The set of all descents of σ is denoted $\mathsf{Des}(\sigma)$.

(a) Let $w_{n,k}$ be the number of pairs (σ, α) where $\sigma \in S_n$ and α is a k-subset of $\mathsf{Des}(\sigma)$. Prove that

$$\sum_{n,k\geq 0} w_{n,k} \frac{x^n t^k}{n!} = \left(1 - \frac{e^{xt}-1}{t}\right)^{-1} .$$

(b) Deduce that the number of permutations in S_n with exactly k descents is

$$n!\,[x^n t^k] \left(1 - \frac{e^{x(t-1)}-1}{t-1}\right)^{-1} .$$

5. [10/8/12 pts.] Throughout this problem, X and Y are assumed to be finite sets with $X \cap Y = \emptyset$.

Let $\mathcal{A}(X,Y)$ be the set of permutations $\sigma : (X \cup Y) \to (X \cup Y)$ with the following two properties:

(I) For all $x \in X$, there exists some $j \geq 1$ such that $\sigma^j(x) \in Y$.

(II) For all $y \in Y$, there exists some $j \geq 1$ such that $\sigma^j(y) \in X$.

Here σ^j denotes the j-th power of σ under composition. In other words, $\mathcal{A}(X,Y)$ is the set of permutations of $X \cup Y$ in which every cycle contains both an element from X and an element from Y. (Example: if $|X| = 1$, $|Y| \geq 1$, $\mathcal{A}(X,Y)$ consists of all cyclic permutations of $X \cup Y$.)

(a) Let $a_{m,n} = \#\mathcal{A}(X,Y)$, if $|X| = m$ and $|Y| = n$. Prove that

$$\sum_{m,n\geq 0} a_{m,n} \frac{x^m y^n}{m!n!} = \frac{(1-x)(1-y)}{1-x-y} ,$$

and hence $a_{m,n} = mn(m+n-2)!$ for $m+n \geq 1$.

(b) Let $p_{m,n}$ denote the number of permutations in $\mathcal{A}(X,Y)$ that have an even number of cycles, if $|X| = m$ and $|Y| = n$, and let $q_{m,n}$ denote the number that have an odd number of cycles. (Example: if $m = 1$, $n \geq 1$, we have $p_{m,n} = 0$ and $q_{m,n} = a_{m,n}$, since every permutation in $\mathcal{A}(X,Y)$ has exactly one cycle.) Prove that for $m, n \geq 1$,

$$q_{m,n} - p_{m,n} = m!n! .$$

(c) Let $\mathcal{B}(X,Y)$ be the set of *all* functions $\sigma : (X \cup Y) \to (X \cup Y)$ (not just the permutations!) that have properties (I) and (II). Let $b_{m,n} = \#\mathcal{B}(X,Y)$, if $|X| = m$ and $|Y| = n$. Prove that for $m+n \geq 1$,

$$b_{m,n} = mn(m+n)^{m+n-2} .$$