ENUMERATION COMPREHENSIVE EXAMINATION

Friday, June 1, 2:00 - 5:00 pm in MC 6486

No additional materials allowed

(1) (a) A *parity-preserving* subset $\{\alpha_1, \ldots, \alpha_k\}$ of $\{1, \ldots, n\}$, where $1 \leq \alpha_1 < \ldots < \alpha_k \leq n$, satisfies the parity condition $\alpha_i \equiv i (\text{mod } 2)$, for $i = 1, \ldots, k$. Let a_n be the number of parity-preserving subsets of $\{1, \ldots, n\}$ (of any cardinality), $n \geq 0$. Prove that

$$\sum_{n \geq 0} a_n x^n = \frac{1+x}{1-x-x^2}.$$

(b) From (a), we have the recurrence $a_n = a_{n-1} + a_{n-2}$, $n \geq 2$, with initial conditions $a_0 = 1$, $a_1 = 2$. Prove this recurrence directly by describing a bijection between parity-preserving subsets.

(c) Let b_n be the number of subsets (of any cardinality) of $\{1, \ldots, n\}$ with no consecutive pairs of integers as elements, $n \geq 0$. Prove that

$$\sum_{n \geq 0} b_n x^n = \frac{1+x}{1-x-x^2}.$$

(d) From parts (a) and (c), we have proved that $a_n = b_n$, for each $n \geq 0$. Prove this equality directly, by describing an explicit bijection between the appropriate sets.

(2) (a) Let

$$G(y, w, x) = \prod_{i \geq 0} \frac{1 - x^i w y}{1 - x^i y}.$$

(i) Prove that

$$G(y, w, x) = 1 + \sum_{m \geq 1} \frac{(1-w)(1-wx)\ldots(1-wx^{m-1})}{(1-x)(1-x^2)\ldots(1-x^m)} y^m.$$

(ii) Deduce from part (i) that

$$\prod_{i=0}^{n-1} \frac{1}{1-x^i y} = 1 + \sum_{m \geq 1} \binom{n-1+m}{m}_x y^m,$$

where

$$\binom{a+b}{b}_x = \frac{(1-x)(1-x^2)\cdots(1-x^{a+b})}{(1-x)(1-x^2)\cdots(1-x^a)(1-x)(1-x^2)\cdots(1-x^b)}.$$

(b) Give a combinatorial proof that

$$\prod_{j=1}^{\infty}(1-x^j) = \sum_{m=-\infty}^{\infty} (-1)^m x^{\frac{m}{2}(3m-1)}.$$

(3) Let $\mathcal{A}$ be the class of rooted labelled trees where each vertex has at most two children. Do not allow the empty tree in your class and do not put any order structure on the children. Let $\mathcal{B}$ be the class of Motzkin paths (lattice paths with unit steps up, down or level, starting and ending at height 0, that are never at negative height) where the up and level steps come in two colours.

(a) Show that the exponential generating function of $\mathcal{A}$ is

$$A(x) = \frac{1-x-\sqrt{1-2x-x^2}}{x}$$

(b) Find an expression for a_n/b_n, $n \geq 1$, where a_n is the number of elements of $\mathcal{A}$ with n vertices and b_n is the number of elements of $\mathcal{B}$ with n steps.

(c) Give a bijective proof of the expression you found in the previous part.

(4) (a) Let $w = t(1+w)^z$, and

$$G(t) = 1 + \sum_{k\geq 1} \frac{x}{k}\binom{x-1+kz}{k-1} t^k.$$

Prove that $G(t) = (1+w)^x$.

(b) For sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$, prove that

$$a_n = b_n + \sum_{k=1}^{n} \frac{x}{k}\binom{x-1+kz}{k-1} b_{n-k}, \qquad n \geq 0,$$

if and only if

$$b_n = a_n - \sum_{k=1}^{n} \frac{x}{k}\binom{-x-1+kz}{k-1} a_{n-k}, \qquad n \geq 0.$$

(5) (a) Let $s(n,k)$ denote the number of permutations of $\{1,\ldots,n\}$ with k cycles, and let

$$\mathbf{s}(t,u) = \sum_{n\geq 0}\sum_{k\geq 0} s(n,k) u^k \frac{t^n}{n!}.$$

Prove that

$$\mathbf{s}(t,u) = (1-t)^{-u}.$$

(b) Determine the average number of cycles among all permutations of $\{1, \ldots, n\}$, $n \geq 1$.

(c) Let $S(n, k)$ denote the number of partitions of $\{1, \ldots, n\}$ into k nonempty disjoint subsets, and let

$$\mathbf{S}(t, u) = \sum_{n \geq 0} \sum_{k \geq 0} S(n, k) u^k \frac{t^n}{n!}.$$

Prove that

$$\mathbf{S}(t, u) = \exp\big(u(e^t - 1)\big).$$

(d) Prove that

$$\sum_{k=m}^{n} (-1)^{n-k} S(n, k)\, s(k, m) = \delta_{n,m}, \qquad n \geq m \geq 0,$$

where $\delta_{n,m} = 1$, for $n = m$, and $\delta_{n,m} = 0$, for $n \neq m$.