C&O — CONTINUOUS OPTIMIZATION

COMPREHENSIVE EXAM — Summer 2016

Monday, June 13, 2016, 1:30-4:30PM

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Contents

1 Linear Programming and Complementarity

Let $A \in \mathbb{R}^{m\times n}$ be a given matrix, and $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$ given vectors. Let (P), (D) stand for the following pair of primal-dual linear programming problems:

$$(P)\quad \begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{array} \qquad\qquad (D)\quad \begin{array}{ll} \max & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & A^T\mathbf{y} + \mathbf{s} = \mathbf{c}, \\ & \mathbf{s} \geq \mathbf{0}. \end{array}$$

Recall that we say that a pair of feasible points $(\mathbf{x}, (\mathbf{y}, \mathbf{s}))$ is *complementary* if $\mathbf{x}^T\mathbf{s} = \mathbf{0}$.

1. State a theorem that relates complementarity to optimality.
2. A pair of feasible points is said to be *strictly complementary* if for each $i = 1, \dots, n$, $x_i s_i = 0$ and $x_i + s_i > 0$. Argue that if $(\mathbf{x}^*, (\mathbf{y}^*, \mathbf{s}^*))$ is a feasible strictly complementary solution, then for any optimizer $\hat{\mathbf{x}}$ of the primal, $\text{supp}(\hat{\mathbf{x}}) \subset \text{supp}(\mathbf{x}^*)$. Here, "$\text{supp}(\mathbf{x})$" denotes the indices (i.e., a subset of $\{1, \dots, n\}$) of nonzero entries of $\mathbf{x}$. [Hint: $\hat{\mathbf{x}}$ must also be complementary with $(\mathbf{y}^*, \mathbf{s}^*)$.]
3. Say that $B \cup N$ (both B, N are subsets of $\{1, \dots, n\}$) is a *strict complementarity partition* of $\{1, \dots, n\}$ for the above LP if there is a strictly complementary solution

such that $\text{supp}(\mathbf{x}^*) = B$ while $\text{supp}(\mathbf{s}^*) = N$. Show that the strict complementarity partition is uniquely determined by the LP, i.e., it cannot have two distinct strict complementarity partitions.

2 Convex Functions

Say that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is *strongly convex* with *modulus* $\mu > 0$ if the function $g(\mathbf{x}) \equiv f(\mathbf{x}) - \frac{\mu}{2}\|\mathbf{x}\|^2$ is convex.

1. Show that an equivalent definition is: for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\lambda \in [0, 1]$,

$$f((1-\lambda)\mathbf{x} + \lambda\mathbf{y}) \leq (1-\lambda)f(\mathbf{x}) + \lambda f(\mathbf{y}) - \frac{1}{2}\mu\lambda(1-\lambda)\|\mathbf{x} - \mathbf{y}\|^2.$$

   [Hint: write $\|(1-\lambda)\mathbf{x} + \lambda\mathbf{y}\|^2$ in terms of λ and $\|\mathbf{x}\|^2$, $\|\mathbf{y}\|^2$ and $\|\mathbf{x} - \mathbf{y}\|^2$.]

2. Suppose also that f is differentiable. Show that strong convexity implies that

$$(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y}))^T(\mathbf{x} - \mathbf{y}) \geq \mu\|\mathbf{x} - \mathbf{y}\|^2,$$

   for all $\mathbf{x}, \mathbf{y}$.

3. Consider the function $f(\mathbf{x}) = \|A\mathbf{x} - \mathbf{b}\|^2$, where $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Suppose that $\text{rank}\,(A) = n$. Show that f is strongly convex.

3 Second Order Cone

The *second-order cone* C_2^n is defined to be

$$C_2^n = \left\{\mathbf{x} \in \mathbb{R}^n \,:\, x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2}\right\}.$$

Second-order cone programming (SOCP) in standard form means minimizing $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \in C_2^{n_1} \times \cdots \times C_2^{n_r}$, where $\mathbf{c}$ is a given n-vector, A is a given $m \times n$ matrix, $\mathbf{b}$ is a given m-vector, and $n_1 + \cdots + n_r = n$ so that the containment makes sense.

1. Show that the second-order cone is a convex set.

2. Define the following function Φ that maps $\mathbb{R}^n$ (vectors) into S^n ($n \times n$ symmetric matrices):

$$\Phi(\mathbf{x}) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_1 & & \\ \vdots & & \ddots & \\ x_n & & & x_1 \end{bmatrix}.$$

Entries of $\Phi(\mathbf{x})$ outside the main diagonal and first row and column are zeros.

Show that $\mathbf{x} \in C_2^n \iff \Phi(\mathbf{x}) \succeq 0$. [Hint: One possible approach is to consider products of the form $\mathbf{v}^T\Phi(\mathbf{x})\mathbf{v}$; use a carefully chosen $\mathbf{v}$ for one direction and the Cauchy-Schwarz inequality for the other. Another possible approach is to use a characterization of positive definiteness in terms of Cholesky factorization applied to a reordering of $\Phi(\mathbf{x})$.]

3. The result of Item 2 is usually cited to justify the statement that: *second-order cone programming is a special case of semidefinite programming*. Explain.

4 Optimality Conditions

For this question, assume that $f, g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$, are convex differentiable functions and that $\bar{x} \in \Omega \subseteq \mathbb{R}^n$; Ω a convex set.

1. (a) Define the tangent cone of Ω at $\bar{x}$, and denote it by $T_\Omega(\bar{x})$. [Hint: You can take advantage of the convexity of Ω.]
   (b) Define the nonnegative polar of a convex cone K and denote it by K^*.

2. State and prove a *characterization of optimality* for
$$\bar{x} \in \operatorname*{argmin}_{x\in\Omega} f(x).$$
(The necessity part for general functions f and general sets Ω is sometimes called the Rockafellar-Pshnenichnyi condition.)

3. Consider the following constrained convex optimization problem with $\Omega = \mathbb{R}^n$.
$$(CP) \qquad \begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & g_i(x) \le 0, \quad i = 1, \ldots, m. \end{array}$$
   (a) State the Lagrangian function $L(x, \lambda)$ for the constrained problem (CP).
   (b) Define the linearizing cone for the constraints in (CP) at a feasible point $\bar{x}$. Find and prove the relationship between this linearizing cone and the tangent cone of the feasible set, i.e., a subset condition.
   (c) State the Karush-Kuhn-Tucker optimality conditions and prove it under a *weakest constraint qualification* obtained from the cone conditions in Item 3b, i.e., that the subset condition holds with equality.

5 Duality and Supporting Hyperplanes

Let the f, g_i and set Ω be as above in Section 4. Consider the above constrained convex program (CP) with the additional set constraint
$$(CPS) \qquad \begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & g_i(x) \le 0, \quad i = 1, \ldots, m \\ & x \in \Omega. \end{array}$$

1. State the Slater condition first for (CP) and then for (CPS).

2. State the Lagrangian dual of (CPS). Prove that weak duality holds.

3. Consider the convex program

$$(CPSS) \qquad \begin{array}{ll} \min_x & f(x) := e^{-\sqrt{x_1 x_2}} \\ \text{s.t.} & x_1 = 0, \\ & x \in \Omega := \mathbb{R}^2_+. \end{array}$$

   Here f is defined as $+\infty$ outside of Ω.

   (a) Derive the Lagrangian dual of (CPSS).

   (b) Find the optimal primal and dual values. Explain your results.

6 Constrained Optimization

Let

$$f : \mathbb{R}^n \to \mathbb{R}, \; g : \mathbb{R}^n \to \mathbb{R}^m, \; h : \mathbb{R}^n \to \mathbb{R}^p$$

be differentiable functions. Consider the general constrained nonlinear program

$$(NLP) \qquad \begin{array}{ll} \min_x & f(x) \\ \text{s.t.} & g(x) \leq 0 \\ & h(x) = 0. \end{array}$$

1. State the Mangasarian-Fromovitz constraint qualification for (NLP) at a feasible point $\bar{x}$.

2. State the appropriate KKT conditions under a constraint qualification at a feasible point $\bar{x}$.

3. Consider the above (NLP) with only equality constraints $h(x) = 0$. (There are no inequality constraints.)

   (a) Construct an augmented Lagrangian for the nonlinear program (NLP).

   (b) Relate an **unconstrained** stationary point of the augmented Lagrangian to the original nonlinear program.

   (c) Give one step of the augmented Lagrangian algorithm, or any other algorithm that utilizes the augmented Lagrangian formulation.