Department of Combinatorics and Optimization
CONTINUOUS OPTIMIZATION COMPREHENSIVE
Spring 2000: **3 hours**
Examiners: Mike Best and Henry Wolkowicz

Instructions: Answer no more than 5 questions. Questions have equal value. Complete answers are preferred over fragmented ones.

1. Let f be a convex differentiable function of n variables, let A be an (m, n) matrix and b an m -vector. Consider the problem

$$\min\{f(x) \mid Ax \leq b, \}. \tag{1}$$

Let x^* be optimal for (1) and assume the gradients of those constraints which are active at x^* are linearly independent. Prove directly that the Karush-Kuhn-Tucker conditions are satisfied at x^*.

2. Consider the (primal) nonlinear programming problem

$$(NLP) \qquad \begin{array}{rcl} f^* = & \inf & f(x) \\ & \text{subject to} & g_j(x) \leq 0, \quad j = 1, \ldots r, \\ & & x \in X \subset \Re^n, \end{array}$$

where all functions are assumed to be twice differentiable, there exists at least one feasible solution, and the optimal value is bounded from below, i.e. $-\infty < f^* < \infty$.

A vector $\mu^* = (\mu_1, \ldots, \mu_r)$ is said to be a *Lagrange multiplier vector* (or simply a *Lagrange multiplier*) for NLP if

$$\mu_j^* \geq 0, \quad j = 1, \ldots, r,$$

and

$$f^* = \inf_{x \in X} L(x, \mu^*).$$

The *dual function* is

$$q(\mu) = \inf_{x \in X} L(x, \mu).$$

The *domain of* q is the set where q is finite

$$D = \{\mu : q(\mu) > -\infty\}.$$

The *Lagrangian dual problem* is

$$q^* = \sup_{\mu \geq 0} q(\mu).$$

Prove the following:

(a) The domain D of the dual function q is convex and Q is concave over D.

(b) Weak duality holds.

(c) If there is no duality gap ($q^* = d^*$), then the set of Lagrange multipliers is equal to the set of optimal dual solutions.
While, if there is a duality gap, then the set of Lagrange multipliers is empty.

3. Under the assumptions and definitions of Problem 2, prove the following:

   (a) (x^*, μ^*) is an optimal solution-Lagrange multiplier pair if and only if

$$x^* \in X, \quad g(x^*) \leq 0, \qquad \text{(Primal Feasibility)} \tag{2}$$

$$\mu^* \geq 0, \qquad \text{(Dual Feasibility)} \tag{3}$$

$$x^* \in \arg\min_{x \in X} L(x, \mu^*), \qquad \text{(Lagrangian Optimality)} \tag{4}$$

$$\langle \mu^*, g(x^*) \rangle = 0, \qquad \text{(Complementary Slackness)} \tag{5}$$

   (b) (x^*, μ^*) is an optimal solution-Lagrange multiplier pair if and only if $x^* \in X, \mu^* \geq 0$, and (x^*, μ^*) is a saddle point of the Lagrangian, in the sense that

$$L(x^*, \mu) \leq L(x^*, \mu^*) \leq L(x, \mu^*), \quad \forall x \in X, \mu \geq 0. \tag{6}$$

4. Let f be a convex differentiable function of n variables and let $\nabla f(x)$ denote its gradient. Let A be an (m, n) matrix and b an m -vector. Consider the problem

$$\min\{f(x) \mid Ax = b, \quad x \geq 0\}. \tag{7}$$

   Show that if $y = x^*$ is optimal for the LP

$$\min\{(\nabla f(x^*))'y \mid Ay = b, \quad y \geq 0\}$$

   then x^* solves (7).

5. Consider the equality constrained nonlinear problem

$$(NEP) \qquad \begin{array}{lll} f^* = & \inf & f(x) \\ & \text{subject to} & h_i(x) = 0, \quad i = 1, \ldots m. \\ & & x \in X \subset \Re^n, \end{array}$$

where the functions are assumed to be continuous, X is a closed set, and there exists at least one feasible solution. Define the *augmented Lagrangian function*

$$L_c(x, \lambda) = f(x) + \lambda^t h(x) + \frac{c}{2}||h(x)||^2,$$

where c is a positive penalty parameter. For $k = 0, 1, \ldots,$, let x^k be a global minimum of the problem

$$\begin{array}{lll} f^* = & \min & L_{c^k}(x, \lambda^k) \\ & \text{subject to} & x \in X, \end{array}$$

where $\{\lambda^k\}$ is bounded, $0 < c^k < c^{k+1}$ for all k, and $c^k \to \infty$. Then every limit point of the sequence $\{x^k\}$ is a global minimum of the original problem NEP.

6. Let $a_1, \ldots, m$ be n-vectors, $b_1, \ldots, b_m$ be scalars, $A = [a_1, \ldots, a_m]$, $b = (b_1 \ldots, b_m)'$ and

$$R = \{x \mid Ax \leq b\}.$$

(a) Define an extreme point for R.

(b) Assume $R \neq \emptyset$. Prove R possesses an extreme point if and only if $\text{rank}(A) = n$.