Department of Combinatorics and Optimization

CONTINUOUS OPTIMIZATION COMPREHENSIVE

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Instructions: Answer as many questions as you can. Complete answers are preferred over fragmented ones. Questions have equal value.

1. Find an upper bound, in terms of the problem data, to the maximum value of the following linear program:

$$\max p^T x \text{ subject to } Ax \le b, e^T x \le 1, x \ge 0.$$

Here $p \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathrm{RR}^m$, and e is a vector of ones in $\mathbb{R}^m$.

2. Consider the optimal control problem

$$\begin{array}{rl} \min & \sum_{i=0}^{N-1} |u_i|^2 \\ \text{subject to} & x_{i+1} = A_i x_i + B_i u_i, \quad i = 0, \ldots, N-1, \\ & x_0, \quad \text{given} \\ & x_N \ge c, \end{array}$$

where c is a given vector, and A_i, B_i are matrices of appropriate size. Show that a dual problem is of the form

$$\begin{array}{rl} \min & \mu^T Q \mu + \mu^T d \\ \text{subject to} & \mu \ge 0, \end{array}$$

where Q is an appropriate $n \times n$ matrix (n is the dimension of x_N) and $d \in \mathbb{R}^n$ is an appropriate vector.

3. For the problem $\min\{f(x)\}$, suppose an iterative method is used according to $x_{j+1} = x_j - B_j g_j$, where $g_j = \nabla f(x_j)$ and B_j is a positive definite matrix chosen to approximate $H_j \equiv H_f(x_j)^{-1}$. Assume $\{x_j\}$ converges to z where $\nabla f(z) = 0$. Let $e_j = \|B_j - H_j\|$. Under what conditions on e_j will the rate of convergence of $\{x_j\}$ be

   (a) superlinear,

   (b) quadratic?

   In each case, prove your result and state any differentiability requirements on f.

4. Suppose that for some real number $\gamma > 0$, $x(\gamma) > 0$ solves the interior penalty (barrier) problem

$$\min \left\{ f(x) - \gamma \sum_{j=1}^{n} \log x_j \ \mid \ Ax = b \right\}$$

where $f\colon \mathbb{R}^n \mapsto \mathbb{R}$, A is an mxn real matrix, b is an mx1 real vector and f is convex and differentiable on $\mathbb{R}^n$. Give a lower bound to

$$\inf \{ f(x) \mid Ax = b, \ \ x \geq 0 \}$$

in terms of $f(x(\gamma))$, γ and n. Establish your claim.

5. Suppose the problem

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & h(x) = 0, \end{array} \tag{1}$$

(where $f\colon \mathbb{R}^n \mapsto \mathbb{R}$ and $h\colon \mathbb{R}^n \mapsto \mathbb{R}^m$ are continuous functions) has a solution x^*. Let M be an optimistic estimate of $f(x^*)$, that is, $M \leq f(x^*)$. Consider the unconstrained problem

$$\min_x v(M, x) := (f(x) - M)^2 + \|h(x)^2\|. \tag{2}$$

Consider the following algorithm. Given $M_k \leq f(x^*)$, a solution x_k to problem (2) with $M = M_k$ is found, then M_k is updated by

$$M_{k+1} = M_k + [v(M_k, x_k)]^{1/2} \tag{3}$$

and the process repeated.

(a) Show that if $M = f(x^*)$, a solution of (2) is a solution of (1).
(b) Show that if x_M is a solution of (2), then $f(x_M) \leq f(x^*)$.
(c) Show that if $M_k \leq f(x^*)$ then M_{k+1} determined by (3) satisfies $M_{k+1} \leq f(x^*)$.
(d) Show that $M_k \longrightarrow f(x^*)$.

6. Consider the following bounded-variable LP with a single general linear constraint:

$$\begin{array}{ll} \max_x & \sum_{i=1}^{n} c_i x_i \\ \text{subject to} & \sum_{i=1}^{n} a_i x_i = b \\ & 0 \leq x_i \leq u_i, \quad i = 1, \ldots, n. \end{array}$$

(a) State an LP dual in which there is a dual variable corresponding to the equation and a dual variable for each upper bound constraint.

(b) Assume that $a_i > 0$ and $u_i > 0, i = 1, \ldots, n$, and that the variables have been indexed so that $c_i/a_i \geq c_{i+1}/a_{i+1}, i = 1, \ldots, n-1$. If $\sum_{i=1}^{n-1} a_i u_i + a_n u_n/2 = b$, state primal and dual optimal solutions and verify that objective values of these two solutions are identical. (Hint: The preceeding expression for b determines a primal optimal solution.)