C&O — CONTINUOUS OPTIMIZATION

COMPREHENSIVE EXAM — Summer 2006

MC 5158A, Wednesday, June 7, 2006, 1:00pm – 4:00pm **(3 hours)**

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Try to answer as many questions as you can. Complete answers will be preferred to partial solutions.

1. Let $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$, $h : \mathbb{R}^n \to \mathbb{R}^q$ be given. Consider

$$(P) \qquad \begin{array}{rl} \inf & f(x) \\ \text{subject to:} & g(x) \leq 0 \\ & h(x) = 0. \end{array}$$

   (a) State the *Mangasarian-Fromowitz Constraint Qualification* for (P).

   (b) State the Karush-Kuhn-Tucker theorem for (P).

   (c) Describe the conditions under which (P) becomes a convex optimization problem. Then, state the (corresponding, stronger version of) Karush-Kuhn-Tucker theorem for (P) when it is a convex optimization problem.

   (d) Let $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{q \times n}$, $c \in \mathbb{R}^n$ be given. Complete the following so that it is a theorem of the alternative. Then prove it using the duality theorem of linear programming. Finally, explain its connection to the Karush-Kuhn-Tucker Theorem.

   "Exactly one of the following systems has a solution:

   (I) $\quad \exists d \in \mathbb{R}^n$ such that $Ad \leq 0, Bd = 0, c^T d > 0$;

   (II) $\quad \exists \lambda \in \mathbb{R}^p_+, \mu \in \mathbb{R}^q$ such that ...."

2. Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a function with nonempty domain (i.e., $\{x \in \mathbb{R}^n : f(x) < +\infty\} \neq \emptyset$) and let $S \subseteq \mathbb{R}^n$ be a nonempty set.

   (a) Give the definitions of the following terms:

      i. *Fenchel conjugate* of f;

      ii. *indicator function* and *support function* of S.

   (b) Show that if f is positively homogeneous, then its Fenchel conjugate is the indicator function of a closed convex set.

   (c) Give a necessary and sufficient condition for f to be the support function of a set. Justify your answer.

3. Let $\lambda_{\max}(X)$ be the largest eigenvalue of the real n-by-n symmetric matrix X. Let (P) be the convex optimization problem

$$\begin{aligned} &\inf\ \lambda_{\max}(X) \\ &\text{s.t.}\ \ \operatorname{tr}(A_i X) = b_i\ (1 \le i \le m) \end{aligned}$$

where $A_1, \ldots, A_m$ are real symmetric matrices and b is a real m-vector.

(a) Show that

i. the Fenchel conjugate of $\lambda_{\max}$ under the inner product $(X, Y) \mapsto \operatorname{tr}(XY)$ is the indicator function of the set of symmetric positive semidefinite matrices with unit trace, and

ii. the subdifferential $\partial\lambda_{\max}(X)$ is the convex hull of matrices of the form vv^T, where v is a unit eigenvector associated with the largest eigenvalue of X.

(b) Formulate the Lagrange dual problem of (P) as an inequality-constrained convex optimization problem.

(c) Show that if A_1 is positive definite and (P) has a feasible solution, then (P) has an optimal solution X^*.

(d) Show that if (P) has an optimal solution X^* with n distinct eigenvalues, then there exists a real m-vector μ such that

$$\sum_{i=1}^{m} \mu_i A_i \text{ is positive semidefinite and of rank one.}$$

4. Consider the constrained optimization problem:

$$\begin{aligned} &\inf\ f(x) \\ &\text{s.t.}\ h(x) = 0, \\ &\qquad\quad x \in X, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^m$ are given functions, and $X \subseteq \mathbb{R}^n$ is a given subset.

(a) Write down the augmented Lagrangian function $L_c(x, \lambda)$ parameterized by c.

(b) Suppose that f, h are continuous, X is compact with $\{x \in X : h(x) = 0\}$ nonempty, and that x^* is an optimal solution. For each k, let x_k be a global minimizer of $\min\{L_{c_k}(x, \lambda_k) : x \in X\}$, where $\{\lambda_k\}$ is bounded and $\{c_k\}$ is a strictly increasing sequence diverging to $+\infty$. Show that $\lim_{k\to\infty} f(x_k) = f(x^*)$.

5. Consider the linear programming problem:

$$(P) \qquad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b,\ x \geq 0, \end{array}$$

where A is a real m-by-n matrix of rank m, b is a real m-vector and c is a real n-vector.

(a) Write down

i. the logarithmic barrier for (P), and

ii. the barrier problem (B_μ) parameterized by $\mu > 0$.

(b) Write down the optimality conditions for (B_μ) in a form from which primal-dual Newton directions can be derived.

(c) Write down the linear system of equations that determines the primal-dual Newton directions $(\Delta x, \Delta s)$, and show that the Newton directions are uniquely determined if the current feasible iterates (x, s) are positive.

(d) Let

$$d(x, s; \mu) := \sqrt{\sum_{i=1}^{n} \left(\frac{x_i s_i}{\mu} - 1 \right)^2}.$$

Suppose the current feasible iterates (x, s) are positive. Use the fact that

$$d(x, s; \mu) \leq \frac{1}{2} \implies \frac{1}{\mu} \sqrt{\sum_{i=1}^{n} (\Delta x)_i^2 (\Delta s)_i^2} \leq d(x, s; \mu)^2$$

to show that the iterates $(x + \Delta x, s + \Delta s)$ are positive and

$$d(x + \Delta x, s + \Delta; \mu) \leq d(x, s; \mu)^2$$

whenever $d(x, s; \mu) \leq 1/2$.

(e) Based on the results of part (d), describe a primal-dual path-following algorithm for (P) that generates a sequence $\{x^k, s^k)\}$ of primal-dual feasible solutions whose duality gap $\{(x^k)^T s^k\}$ converges Q-linearly to zero, when started from primal-dual feasible solutions (x^0, s^0) satisfying

$$d(x^0, s^0; \mu^0) \leq \frac{1}{4}$$

where $\mu^0 = (x^0)^T s^0 / n$.