C&O — CONTINUOUS OPTIMIZATION

COMPREHENSIVE EXAM — Summer 2008

Friday, July 11, 2008, 9:00 am to 12 noon (3 hours), MC 5158A

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1. This question concerns the following linear feasibility problem:

   (P): Given an $m \times n$ matrix A, determine whether there exists $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$, and $\mathbf{x} \neq \mathbf{0}$.

   (a) Let $\operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ denote the convex hull of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Consider the problem: given two lists of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_l$ all lying in $\mathbb{R}^p$, determine whether $\operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ and $\operatorname{conv}(\mathbf{w}_1, \ldots, \mathbf{w}_l)$ have a nonempty intersection. Show that this problem can be expressed in the form (P), in which the matrix A is $(p+1) \times (k+l)$.

   (b) Write down a dual feasibility problem (D) for problem (P). It should have the property that the primal is feasible if and only if the dual is infeasible. (Hint: introduce an objective function and use LP duality.)

   (c) Consider the LP problem of maximizing $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} = \mathbf{0}$, $\mathbf{x} \geq \mathbf{b}$ in which $\mathbf{b} < \mathbf{0}$ and $\mathbf{c} > \mathbf{0}$ are chosen arbitrarily. Obviously, this LP is feasible (take $\mathbf{x} = \mathbf{0}$). Show that it is bounded if and only if (P) is infeasible.

2. Let C be a nonempty convex subset of $\mathbb{R}^n$. Let $\mathbf{f} = (f_1, \ldots, f_m)$, where each function $f_i : C \to \mathbb{R}$ is a convex function, and let $g : \mathbb{R}^m \to \mathbb{R}$ be a convex and monotonically nondecreasing function over a convex set that contains the set $\{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in C\}$. "Monotonically nondecreasing" means that for all $\mathbf{u}, \bar{\mathbf{u}}$ in this set such that $\mathbf{u} \leq \bar{\mathbf{u}}$, we have $g(\mathbf{u}) \leq g(\bar{\mathbf{u}})$.

   (a) Show that the function h defined by $h(\mathbf{x}) = g(\mathbf{f}(\mathbf{x}))$ is convex over $C \times \cdots \times C$.

   (b) Show that the function h defined by $h(\mathbf{x}) = \max_{i=1,\ldots,m}\{f_i(\mathbf{x})\}$ is convex over C.

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 objective function, and let $\mathbf{x} \in \mathbb{R}^n$ be a point with a nonvanishing gradient.

   (a) Suppose one wishes to minimize f using the standard (2-norm) trust region method. Write down the optimization problem that defines the trust-region subproblem at current iterate $\mathbf{x}$. Use Δ to denote the radius of the trust region.

   (b) What happens to the solution of the trust region subproblem as the radius Δ shrinks to zero? Explain.

4. (a) Write down the Wolfe conditions for a line search step size in unconstrained optimization.

(b) Consider minimizing the objective function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(\mathbf{x}) = x_1^2 + x_2^2$ starting from $\mathbf{x}_0 = (1, 1)$. Suppose that on each step, a descent direction is generated and suppose the Wolfe conditions are satisfied by the line search on each step. Are these assumptions enough to guarantee convergence to the minimizer at $(0, 0)$? If so, explain why. If not, explain what could go wrong either with an algebraic argument or a careful diagram.

5. Consider the nonlinear unconstrained minimization problem

$$\min_{\mathbf{x}} \sum_{k=1}^{m} \sqrt{\rho^2 + (\mathbf{a}_k^T \mathbf{x} - b_k)^2}, \tag{1}$$

where ρ is a small positive constant. The problem (1) is often taken to be a smooth approximation of the ℓ_1-minimization problem

$$\min \|A\mathbf{x} - \mathbf{b}\|_1.$$

(Here $A \in \mathbb{R}^{m \times n}$ with rows $\mathbf{a}_k^T$.)

(a) Note that (1) is equivalent to the constrained minimization problem

$$\begin{array}{ll} \min & \sum_{k=1}^{m} \sqrt{\rho^2 + y_k^2} \\ \text{s.t.} & \mathbf{y} = A\mathbf{x} - \mathbf{b}, \end{array} \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$. Write down the KKT conditions of (2).

(b) The ℓ_1 minimization problem can be similarly modified by the introduction of $\mathbf{y}$. However, this modification is still not smooth because the objective function involves $|y_k|$. Introduce a second auxiliary vector $\mathbf{w} \in \mathbb{R}^m$ and some additional inequality constraints in order to write the ℓ_1 problem as smooth constrained optimization, and then write its KKT conditions also.

(c) Given a solution to the KKT conditions of (2), show how to obtain multipliers for the ℓ_1 problem that approximately satisfy its KKT conditions. The error in approximation should tend to 0 as $\rho \to 0$. [Hint: take cases based on whether y_i is positive, negative or 0.]

(d) Let p^* be the optimal value of the ℓ_1-minimization problem, and let p^*_{sm} be the optimal value of (1). Show that

$$p^*_{sm} \geq p^* \geq p^*_{sm} - m\rho.$$

[Hint for the second inequality: use a relation among $\sqrt{u^2 + v^2}$, $|u|$ and $|v|$.]