C&O — CONTINUOUS OPTIMIZATION
COMPREHENSIVE EXAM — Summer 2009

MC 2018A, Monday, June 8, 2009, 9:00am – noon (**3 hours**)

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1. Consider the optimization problem

$$\inf\{\mathbf{x}^T Q\mathbf{x} + 2\mathbf{c}^T\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{R}^n\}$$

for given vectors $\mathbf{b}$ in $\mathbb{R}^m$ $\mathbf{c} \in \mathbb{R}^n$ and a given matrices $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = m$, $Q \in \mathbb{R}^{n\times n}$ with Q symmetric positive definite.

   (a) Prove that the problem has a unique optimal solution.

   (b) Find the (Lagrangian) dual problem and solve the dual problem. Using the dual optimal solution, find the primal optimal solution.

   (c) Find the duals of the following problems

$$\inf\{\mathbf{x}^T Q\mathbf{x} + 2\mathbf{c}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbb{R}^n\},$$

   and

$$\inf\{\mathbf{x}^T Q\mathbf{x} + 2\mathbf{c}^T\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{R}^n_+\}.$$

2. (a) Let $C \subset \mathbb{R}^n$ be a nonempty closed convex set and suppose $\hat{\mathbf{x}} \in \mathbb{R}^n \setminus C$. State and prove the *separating hyperplane theorem* for this situation.

   (b) Consider the set C and point $\hat{\mathbf{x}}$ as in part (a). Prove that the infimum of the distances from $\hat{\mathbf{x}}$ to a point in C is equal to the supremum of the distances from $\hat{\mathbf{x}}$ to a hyperplane separating $\hat{\mathbf{x}}$ from C.

   (c) For $K \subseteq \mathbb{R}^n$, define

$$K^* := \left\{\mathbf{y} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{y} \geq 0, \forall \mathbf{x} \in K\right\}.$$

   Suppose $K \neq \emptyset$. Prove or disprove: "K is a closed convex cone iff $K = (K^*)^*$."

3. (a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function. Explain what is meant by a *stationary* (or *critical*) point of f and what is meant by a *local minimizer*.

   (b) State conditions in terms of the first and second derivatives of f that are necessary for $\mathbf{x}^* \in \mathbb{R}^n$ to be a local minimizer. State conditions that are sufficient for local minimality.

   (c) It is possible in principle for the steepest descent method to converge to a stationary point that fails to satisfy the second-order necessary condition, but this almost never happens in practice. The following analysis explains why. Consider minimizing $\mathbf{x}^T D\mathbf{x}$, where D is an $n \times n$ diagonal matrix whose diagonal entries are of mixed signs. Argue that the steepest descent method initiated at a point $\mathbf{x}^0 \neq \mathbf{0}$ will not converge to the stationary point at $\mathbf{x}^* = \mathbf{0}$ except under special circumstances. [Assume that either an exact line search or an inexact line search satisfying the Wolfe conditions is used. Hint: argue that either steepest descent will take an unbounded step or else that certain coordinate entries of $\mathbf{x}$ will grow in magnitude instead of shrinking.]

4. Consider applying Newton's method with a line search $x^{k+1} = x^k - \alpha^k(\nabla^2 f(x^k))^{-1}\nabla f(x^k)$ to minimize a function $f(x)$. In order to achieve asymptotic quadratic convergence, it is necessary for α^k to converge to 1. Determine how fast α^k must converge to 1 (as a function of $\|x^k - x^*\|$) in order to ensure quadratic convergence. An informal Taylor series analysis is acceptable, and you may make all necessary assumptions that usually pertain to Newton's method.

5. Let $S \subseteq \mathbb{R}^n$ an open set, $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$, $h : \mathbb{R}^n \to \mathbb{R}^q$ be given. Consider

$$(P) \qquad \begin{array}{rl} \inf & f(x) \\ \text{subject to:} & g(x) \le 0 \\ & h(x) = 0 \\ & x \in S. \end{array}$$

   (a) State the Karush-Kuhn-Tucker (KKT) theorem for (P) (including all the necessary assumptions on f, g and h).

   (b) Let $e \in \mathbb{R}^n$ denote the vector of all ones, and $A \in \mathbb{R}^{m\times n}$ with $Ae = 0$ be given. Consider the following optimization problem:

$$(P_0) \qquad \begin{array}{rl} \inf & -\ln\left(\prod_{j=1}^{n} x_j\right) \\ \text{subject to:} & Ax = 0 \\ & e^T x = n \\ & x \in \mathbb{R}^n_{++}. \end{array}$$

   Prove that (P_0) has a unique optimal solution.

   (c) State the strongest version of KKT Theorem you can for (P_0).

   (d) What is the unique optimal solution of (P_0)? Prove your claim using the KKT theorem from part (c).

6. (a) Let $D \subseteq \mathbb{R}^n$ be nonempty, open and convex, and $F : D \to \mathbb{R}$ be given such that F is twice continuously differentiable on D and $F(x) > 0$, $\forall x \in D$. Define $f : D \to \mathbb{R}$ by

$$f(x) := \ln(F(x)).$$

   Prove that F is convex on D iff the matrix

$$\nabla^2 f(x) + \nabla f(x)\,[\nabla f(x)]^T$$

   is positive semidefinite for every $x \in D$.

   (b) For $u \in \mathbb{R}^n$, let $U := \mathrm{Diag}(u) \in \mathbb{R}^{n\times n}$. Prove that for every $u \in \mathbb{R}^n$,

$$nU^2 - uu^T \text{ is positive semidefinite.}$$

(c) Let $\mathbf{c} \in \mathbb{R}^n_{++}$, $n \geq 3$. Define

$$F(\mathbf{x}) := \begin{cases} \dfrac{(\mathbf{c}^T\mathbf{x})^{n+1}}{\prod_{j=1}^{n} x_j} & \text{if } \mathbf{x} \in \mathbb{R}^n_{++}, \\ +\infty & \text{otherwise.} \end{cases}$$

Prove that F is convex on $\mathbb{R}^n$. (Hint: It is clear that part (a) is useful here. Part (b) can also be useful; but it may not be as easy to see how...)