C&O — CONTINUOUS OPTIMIZATION

COMPREHENSIVE EXAM — Summer 2025

June 4, 2025, 1:00–4:00 PM (3 hours),

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The exam has 4 problems on 4 pages.
(Please provide careful justification of the answers.)

Contents

1 Unconstrained Minimization

Consider the unconstrained quadratic minimization

$$p^* = \min_{x\in\mathbb{R}^n} q(x) := \frac{1}{2}x^TQx + g^Tx, \tag{1}$$

where $Q \in \mathbb{R}^{n\times n}$, $g \in \mathbb{R}^n$.

1.1 Quadratic Program

1. Show that, without loss of generality, we can assume Q is symmetric, i.e., $Q \in \mathbb{S}^n$. (And assume so below.)

2. Show that the following four conditions are equivalent:

   (a) p^* is finite (is in $\mathbb{R}$).

   (b) There exists $\bar{x} \in \mathbb{R}^n$ that is a local minimum of q. (Define the term *local minimum* before using it.)

(c) There exists $x^* \in \mathbb{R}^n$ that is a global minimum of q. (Define the term *global minimum* before using it.)

(d) $g \in \mathcal{R}(Q)$, the range of Q.

3. Do the four equivalences in Item 2 hold if Q is *not* symmetrized first? (Provide all the details.)

1.2 Trust Region Method for Unconstrained Minimization

Now consider the unconstrained minimization

$$(Unc) \qquad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in \mathbb{R}^n , \end{array} \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is sufficiently smooth (differentiable). In addition, consider the quadratic constrained version of (1), the trust region subproblem, TRS,

$$(TRS) \qquad \begin{array}{rll} p^* = & \min & q(x) = \frac{1}{2}x^T Q x + g^T x \\ & \text{s.t.} & \|x\|^2 \leq \delta^2, \end{array} \tag{3}$$

where $\delta > 0$ is the trust region radius.

1. Outline the *trust region method* for minimization of f. In particular, describe how the TRS arises and the role of δ.

2. Prove that p^* is finite (is in $\mathbb{R}$).

3. State the dual functional and the Lagrangian dual of TRS. Let d^* be the dual optimal value.

4. Prove weak duality for TRS. (Define it first.)

5. Prove that strong duality for TRS holds. (Define it first.)

6. Does strong duality always hold if we allow $\delta = 0$?

2 Interior Point Methods

Consider the standard form linear program, (LP),

$$(LP) \qquad \begin{array}{rll} p^* = & \min & c^T x \\ & \text{s.t} & Ax = b \in \mathbb{R}^m \\ & & x \in \mathbb{R}^n_+ , \end{array}$$

with appropriate data A, b, c. Assume that $p^* \in \mathbb{R}$.

1. Derive the dual program (DP) for (LP).

2. State the optimality conditions for (LP), i.e., use the primal and dual feasibility and complementary slackness.

3. Use the above optimality conditions to derive a primal-dual interior point method for solving (LP). Include the derivation of the normal equations for finding the Newton direction.

4. Can you find an example of (LP) where strict complementarity fails? Why or why not? (Define strict complementarity first.)

5. Can you find an example of a semidefinite program where strict complementarity fails? If you cannot, then can you find an example for a general nonlinear program?

3 Convexity and Coercivity

Let $h : E \to (-\infty, \infty]$ be an extended real-valued function on the Euclidean space E.

1. Define:
   (i) the epigraph of h, $\operatorname{epi} h$;
   (ii) h is convex;
   (iii) h is lower semicontinuous;
   (iv) h is coercive.

2. Prove that h is convex if, and only if, $\operatorname{epi} h$ is convex.

3. Hence, or otherise, prove that the supremum of convex functions is convex. (You may use the fact that the intersection of convex sets is a convex set).

4. Check whether or not the following functions are coercive:

   (a) $f : \mathbb{R}^3 \to \mathbb{R} : (x, y, z) \mapsto |x| + y^2 + z^4$.

   (b) $g : \mathbb{R}^3 \to \mathbb{R} : (x, y, z) \mapsto x^2 + 3xy + y^2 + |z|$.

5. Suppose that h is coercive, lower semicontimnuous and proper. Show that h has a global minimizer over E.

6. Suppose that h is convex and proper. Show that every local minimizer of h is a global minimizer. (h need not be smooth).

4 Subdifferentials and Proximal Point Mappings

Let $h : E \to (-\infty, \infty]$ be an extended real-valued function on the Euclidean space E and let C be a nonempty closed subset of E.

1. Define:
   (i) the subdifferential of h at $x \in E$, $\partial h(x)$;
   (ii) the proximal point mapping of h at $x \in E$, $\operatorname{Prox}_h(x)$;
   (iii) the indicator function of a set C, ι_C;
   (iv) the set C is convex;
   (v) the normal cone operator of C, N_C.
   (vi) the orthogonal projection onto the set C, P_C.

2. Prove that $\partial \iota_C = N_C$.

3. Suppose that C is convex and let $x \in E$. Prove that $N_C(x) = \{0\}$ if, and only if, $x \in \operatorname{int} C$.

4. Suppose that $C = \{(x, y) \in \mathbb{R}^2 \mid ||(x, y) - (2, 1)|| \leq 3\}$. Find N_C at any point in $\mathbb{R}^2$.

5. Prove that $\operatorname{Prox}_h = (\operatorname{Id} + \partial h)^{-1}$, where $\operatorname{Id} : E \to E : x \mapsto x$.

6. Suppose that C is convex and let $x \in E$. Prove that $\operatorname{Prox}_h = P_C$.