Department of Combinatorics and Optimization
CONTINUOUS OPTIMIZATION COMPREHENSIVE
July 2001: **3 hours**
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Instructions: Answer as many questions as you can. Complete answers are preferred over fragmented ones. Questions have equal value.

1. Consider the LP

$$\min\{c'x \mid a_i'x \le b_i,\ i = 1, \ldots, m\},$$

where c and $a_1, \ldots, a_m$ are n-vectors.

(a) State the Karush-Kuhn-Tucker conditions for this problem.

(b) Give a constructive proof that these conditions are necessary for optimality provided the gradients of those constraints active at an optimal solution are linearly independent.

(c) Show geometrically what problems can occur when the linear independence assumption is not satisfied.

2. Given any $m \times n$ matrix A, consider the optimization problem

$$\alpha = \sup\{x^T Ay : ||x||^2 = 1, ||y||^2 = 1\} \tag{1}$$

and the matrix

$$\tilde{A} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}.$$

(a) If μ is an eigenvalue of $\tilde{A}$, prove $-\mu$ is also.

(b) If μ is a nonzero eigenvalue of $\tilde{A}$, use a corresponding eigenvector to construct a feasible solution to problem (1) with objective value μ.

(c) Deduce $\alpha \geq \lambda_{max}(\tilde{A})$ (the largest eigenvalue).

(d) Use the Karush-Kuhn-Tucker theorem to prove any optimal solution of problem (1) corresponds to an eigenvector of $\tilde{A}$. (State carefully the: **theorem, assumptions, constraint qualifications**, that you use, i.e. justify the use of the theorem.)

(e) Deduce $\alpha = \lambda_{max}(\tilde{A})$. (This number is called the *largest singular value of A*.)

3. Consider the convex program

$$(CP) \qquad \begin{array}{ll} \min & f(x) \\ \text{subject to} & g(x) \leq 0, \end{array}$$

where $f: \Re^n \to \Re$ and $g: \Re^n \to \Re^m$ are differentiable and convex functions on $\Re^n$.

(a) State Slater's constraint qualification (CQ) for (CP).

(b) Suppose that Slater's CQ holds and that

$$x(\alpha) = \arg\min_x \{f(x) - \alpha \sum_{j=1}^{n} \log(-g_j(x)) \mid g(x) < 0\}$$

for $\alpha > 0$. What should α satisfy in order that

$$f(x(\alpha)) - \inf_x \{f(x) \mid g(x) \leq 0\} \leq 10^{-6}$$

Prove your claim.

(**Hint:** Recall that the primal-dual interior-point approach for convex programming is similar to that for linear programming.)

4. Let f be a convex function from $\Re^n$ to $\Re$. Suppose that for each nonzero y in $\Re^n$ there exists some positive η such that $f(\eta y) > f(0)$.

(a) Prove that the set

$$L(1) = \{ x \in \Re^n \mid f(x) \leq 1 \}$$

is compact.

(b) Can you draw any conclusion about the compactness of

$$L(n) = \{ x \in \Re^n \mid f(x) \leq n \},$$

where n is an arbitrarily large natural number? Explain.

5. Let f be a convex differentiable function of n variables, let A be an (m, n) matrix and b an m -vector. Consider the problem

$$\min\{f(x) \mid Ax \leq b\}. \tag{2}$$

Let x^* be optimal for (2). Consider the following solution algorithm for (2). Let x_0 be an arbitrary feasible point for (2) and let y_0 be optimal for the LP

$$\min\{\nabla f(x_0)'y \mid Ay \leq b\}. \tag{3}$$

Let

$$\sigma_0 = \arg\min\{f(x_0 + \sigma(y_0 - x_0)) \mid 0 \leq \sigma \leq 1\}.$$

Set $x_1 = x_0 + \sigma_0(y_0 - x_0)$. Replace x_0 with x_1 and repeat.

(a) Sketch the progress of this algorithm.

(b) Would this algorithm be computationally efficient? Why?

(c) Verify the bound

$$f(x^*) \geq f(x_0) + \nabla f(x_0)'(y_0 - x_0).$$

6. (a) Prove that any local minimum of a convex set is also a global minimum. Do *not* make any differentiability assumptions,

   (b) i. State the two term Taylor's series for a twice differentiable function $f(x)$ (x is an n-vector) about a point x_0.

   ii. By taking gradients of both sides of this Taylor's series, can one conclude that

$$\nabla f(x) = \nabla f(x_0) + H(\xi)(x - x_0),$$

where ξ lies on the line segment joining x and x_0?

7. Let $f : \Re^n \to \Re$ be a differentiable convex function on $\Re^n$ and let $x_1, x_2, \ldots, x_k$ be k points in $\Re^n$ such that for some $u \in \Re^k$:

$$\sum_{i=1}^{k} u_i \nabla f(x_i) = 0, \quad \sum_{i=1}^{k} u_i = 1, \quad u \geq 0.$$

Derive a lower bound for $\inf_{x \in \Re^n} f(x)$ in terms of $x_1, x_2, \ldots, x_k$ and u.