On the Birkhoff-Lewis Equations

Daniel Marcotte

Abstract

Introduced by Birkhoff and Lewis in 1946 [4] and formalized by Tutte in 1991 [10], the Birkhoff-Lewis Equations give surprising relations between some specialized chromatic polynomials of planar graphs. This essay fuses the applied, computational approach of Birkhoff and Lewis with the abstract, existence approach of Tutte in order to present a clear introduction to these equations and their uses. We will discuss the derivation of these equations and explore a number of different forms they can take.

1 Reducibility and the Four Colour Theorem

We motivate our discussion with the same topic that motivated the creation of the Birkhoff-Lewis Equations: The Four Colour Theorem.

Definition. A **colouring** of a graph G is an assignment of colours to the vertices where no two adjacent vertices receive the same colour. A colouring that chooses its colours from a set of λ colours is called a λ**-colouring**.

The most famous theorem in graph colouring is the Four Colour Theorem:

Theorem 1 (4CT)**.** *Any planar graph can be 4-coloured.*

We are going to study the Birkhoff-Lewis equations in the context of something called *reducibility*, with an eye towards the proof of the 4CT. Both Appel and Haken's original proof (see [1] and [2]) and Robertson, Saunders, Seymour and Thomas' revised proof ([9]) use the same basic approach to proving the 4CT, with the key tool being this concept of reducibility. So what exactly are we "reducing"?

Definition. Call a planar graph G a **minimal counter-example** if every graph with fewer vertices than G is four colourable, but G itself needs at least five colours. Since we measure size in terms of vertices, we may assume G is a triangulation (i.e., we can always drop all possible edges into a minimal counter-example to obtain a triangulated minimal counter-example).

The approach used in the proofs mentioned above prove the 4CT by proving there is no minimal counter-example (and hence *no* counter-example).

Given a planar graph G, suppose we replace some subgraph of G, say H, with a smaller graph H' to form G'. Suppose further that we prove that G having no 4-colouring implies G' has no 4-colouring. We are now certain that G is not a minimal counter-example. We have *reduced* G to a smaller counter-example. We say that G is **reducible** and we call H a **reducible configuration**. Note that defined this way, any graph which is reducible (i.e., contains a reducible configuration) cannot be a minimal counter-example. Also note that we use the term *configuration* here much more loosely than in the proof of the 4CT. For our purposes this is sufficient, and has the benefit of letting our terminology line up better with the actual 4CT proof.

In a nutshell, both proofs of the four colour theorem prove that every potential minimal counter-example contains a reducible configuration. We get our hands on potential minimal counter-examples through *connectivity*:

Definition. A graph G is k**-connected** if deleting any set of fewer than k vertices of G results in a connected graph. We call a set of l vertices that result in a graph which is not connected an l**-separation**.

We have the following property:

Lemma 2 (See [3]). *Any minimal counter-example G is 5-connected, and the only 5-separations are the five neighbours of a vertex of degree 5. We sum up these two properties by saying G is* **internally 6-connected**.

If we could show every internally 6-connected graph contained a reducible configuration, we would be in possession of a proof for the 4CT.

Definition. Call a set U of configurations **unavoidable** if every internally 6-connected graph contains an element of U as a subgraph.

Sketch of the 4CT proof.

1. Define a set U of configurations and prove it is unavoidable.

2. Prove that every element of U is reducible.

Naturally this is an outrageous abstraction of the actual proof. Even in the "simplified" proof of Robertson et al. there are 633 configurations in U. The question of how to find U—and how to prove its unavoidability—is addressed in appendix A.

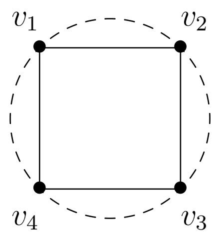
So what are the Birkhoff-Lewis equations, and how do they relate to reducibility? Consider the following approach to reducibility:

Suppose we have a minimal counter-example G. If we could prove G is reducible, having assumed nothing more than that it is a minimal counter-example, we would have a contradiction which proves the 4CT. We could—if we were so inclined—take a separating cycle $\mathcal{C}$ and note that the subgraph H_1 obtained by deleting everything outside $\mathcal{C}$ is four-colourable by our assumption that G is a minimal counter-example. Similarly, the subgraph H_2 obtained by deleting everything inside G is four-colourable.

Knowing nearly nothing about G (except that it is a minimal counter-example with a separating cycle $\mathcal{C}$), we can say for sure that any two colourings of H_1 and H_2 disagree on how they colour the vertices in $\mathcal{C}$—otherwise we could just glue those two colourings together to obtain a colouring of G. Since colourings can be permuted, the specific colours involved are not important to this gluing process; what matters is whether the two colourings partition the vertices of $\mathcal{C}$ the same way.

The Birkhoff-Lewis equations relate to counting such colouring partitions.

To elaborate, consider the case where $\mathcal{C}$ has length 4. Figure 1 lists the possible partitions that can be induced by 4-colourings of the inside or outside of $\mathcal{C}$.



$$X_1 = (\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\})$$
$$X_2 = (\{v_1, v_3\}, \{v_2\}, \{v_4\})$$
$$X_3 = (\{v_2, v_4\}, \{v_1\}, \{v_3\})$$
$$X_4 = (\{v_1, v_3\}, \{v_2, v_4\})$$

Fig 1. Partitions of $\mathcal{C}$ induced by 4-colourings

Let K_i be the number of ways to 4-colour a graph bounded by $\mathcal{C}$ in such a way that partitions the vertices of $\mathcal{C}$ into X_i. We will prove later that the equation

$$K_1 - K_2 - K_3 + 2K_4 = 0, \tag{1}$$

holds for any planar graph bounded by $\mathcal{C}$, and hence must hold for the inside and outside of $\mathcal{C}$ in G. This is a **Birkhoff-Lewis equation**—an invariant relation between numbers of partition inducing colourings of a cycle $\mathcal{C}$ bounding an embedding of a planar graph G. We use this Birkhoff-Lewis equation to prove the following reducibility result:

Proposition 1. *A minimal counter-example does not have a 4-separation.*

Proof. Suppose to the contrary we have a minimal counter-example G with a 4-separation. We know that G is a triangulation, so our 4-separation forms a cycle $\mathcal{C}$. Assume G is embedded in the plane. Let H^{in} be the subgraph of G obtained by deleting all the vertices embedded outside $\mathcal{C}$. Define H^{out} analogously.

Let K_i^{in} and K_i^{out} be K_i of equation (1) for H^{in} and H^{out} respectively. By our hypothesis that G is a minimal counter-example, both H^{in} and H^{out} are 4-colourable. In terms of our K_i's, this means

$$K_1^{in} + K_2^{in} + K_3^{in} + K_4^{in} \geq 1 \tag{2}$$

$$K_1^{out} + K_2^{out} + K_3^{out} + K_4^{out} \geq 1. \tag{3}$$

We must also have

$$K_i^{in} K_i^{out} = 0, \; i = 1, \ldots, 4 \tag{4}$$

else we could glue together two colourings to obtain a 4-colouring of G. Furthermore, equation (1) tells us that if three K_i's are zero for either H^{in} or H^{out}, then they all are. Together with (2),(3) and (4), this implies that exactly two K_i's are zero for both H^{in} and H^{out}.

Since the roles of "in" and "out" are completely interchangeable (by stereographic projection for example) we suppose *wlog* that $K_1^{out} = 0$. Since the K_i's are nonnegative, $K_4^{out} \neq 0$ for otherwise this would make all of the K_i's for H^{out} zero, contradicting (3). We also suppose *wlog* that $K_3^{out} = 0$ since the proof is completely symmetric in the case where $K_2^{out} = 0$.

Now we have that the K_i's for H^{out} relating to

$$X_1 = \{1, 2, 3, 4\} \tag{5}$$

$$X_3 = \{24, 1, 3\} \tag{6}$$

are zero. This implies that there are no 4-colourings of H^{out} that assign v_1 and v_3 different colours (note that they are in the same cell in both X_2 and X_4). Hence if we replace H^{in} in G with

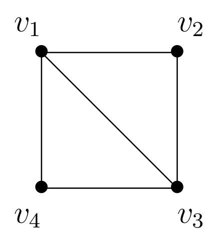


Fig 2. A replacement for H^{in}

we obtain a smaller graph with no 4-colourings. Contradiction. □

So how do we go about obtaining equations such as (1)? In fact, why should (1) hold for all graphs? And can we find similar equations for graphs embedded in cycles of any length? Yes. There is some work to be done though—we cannot even begin to explore relationships between such K_i's until we lay the groundwork for counting colourings: *the chromatic polynomial.*

2 Chromatic Polynomials

If investigating the existence of a λ-colouring in a graph G is a natural question, another natural, more general, question might be *how many λ-colourings of G are there?* If the answer to this more general question is nonzero, then the answer to the existence question is yes. One of the tools used to take this approach is the following:

Definition. Let $P(G;\lambda)$ be the number of λ-colourings of G. We call $P(G;\lambda)$ the **chromatic polynomial** (or **chromial**) of G, a name which is justified below in Proposition 3.

The canonical first examples of chromatic polynomials are the empty graph $\overline{K}_n$ and the complete graph K_n.

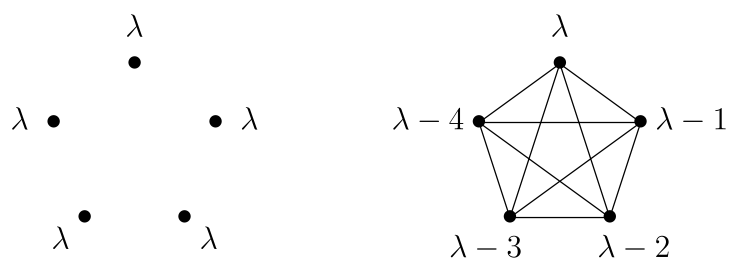


Fig 3. Counting colourings of $\overline{K}_5$ and K_5

In $\overline{K}_n$, we have λ choices of colours for each vertex with no restrictions since there are no edges, hence

$$P(\overline{K}_n, \lambda) = \lambda^n.$$

Now, define $\lambda_{[k]} = \lambda(\lambda-1)(\lambda-2)\dots(\lambda-k+1)$. Then, for K_n, each colour assigned to a vertex leaves one less colour available to all the other vertices, hence

$$P(K_n, \lambda) = \lambda_{[n]}.$$

Normally we cannot look at a graph and easily determine its chromatic polynomial. Deciding whether or not a given graph G admits a λ-colouring is in general $\mathcal{NP}$-Complete, so computing the number of λ-colourings is obviously hard. We are not however entirely without tools.

Definition. We use $G\backslash e$ to denote the graph obtained by **deleting** edge e in G and G/e to denote the graph obtained by **contracting** edge e in G.

Proposition 2 (Chromatic recurrence)**.**

$$P(G; \lambda) = P(G\backslash e; \lambda) - P(G/e; \lambda).$$

Proof. A λ-colouring of $G\backslash e$ either assigns the endpoint of e different colours, or assigns the endpoints of e the same colours. Since a λ-colouring of $G\backslash e$ assigns the endpoints of e different colours if and only if it is a λ-colouring of G, the number of such colourings is $P(G; \lambda)$.

Similarly, a λ-colouring of $G\backslash e$ assigns the endpoints of e the same colours if and only if it is a λ-colouring of G/e, hence the number of such colourings is $P(G/e; \lambda)$. We now have

$$P(G\backslash e; \lambda) = P(G; \lambda) + P(G/e; \lambda),$$

which, after rearrangement, yields the result. □

We compute $P(C_4;\lambda)$ here as an example. By symmetry, no matter which edge of C_4 we choose, deleting it yields P_4, the path on four vertices, and contracting it yields K_3, a triangle.

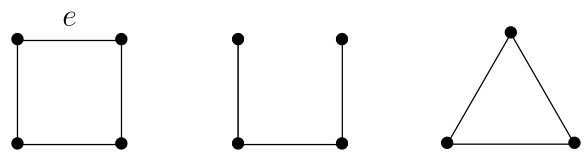


Fig 4. Elements of a chromatic recurrence.

A little thought shows that $P(P_4;\lambda) = \lambda(\lambda-1)^3$, and our work above for complete graphs gives $P(K_3;\lambda) = \lambda(\lambda-1)(\lambda-2)$. By Proposition 2 we have

$$\begin{aligned} P(C_4;\lambda) &= P(P_4;\lambda) - P(K_3;\lambda) \\ &= \lambda(\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2) \\ &= \lambda(\lambda-1)(\lambda^2-3\lambda+3). \end{aligned}$$

We close up this introduction to chromatic polynomials by proving they are, in fact, polynomials. A slightly deeper, though still accessible, treatment of chromatic polynomials can be found in [12].

Proposition 3. *For any graph G, $P(G;\lambda)$ is a polynomial in λ.*

Proof. We proceed by induction on the number m of edges of G. For $m=0$, we have $P(G;\lambda) = P(\overline{K}_n;\lambda) = \lambda^n$. For $m \geq 1$, we can choose an edge e to contract and delete to obtain

$$P(G;\lambda) = P(G\backslash e;\lambda) - P(G/e;\lambda). \tag{7}$$

by Theorem 2. By induction both terms on the right of (7) are polynomials, and hence their difference is too. □

3 The Birkhoff-Lewis Equations

We now derive the Birkhoff-Lewis equations following Tutte's work in both [10] and [11]. We generalize the bounding cycle $\mathcal{C}$ from the second section in the following manner: We consider a graph G embedded in the plane, bounded by a circle C and let S_n denote the vertices $\{v_1, v_2, \ldots, v_n\}$ of G on C, which we assume are indexed in cyclic order. The Birkhoff-Lewis equations are then built out of special chromatic polynomials of G which depend on partitions of S_n.

Definition. A **partition** X of S_n is a set of non-null disjoint sets (called **parts**) whose union is S_n.

For example, suppose we have a planar embedding of a graph G bounded by a circle C, and five vertices of G lie on C. We would then enumerate the five vertices $\{v_1, v_2, v_3, v_4, v_5\}$ in cyclic order and let them be S_5.

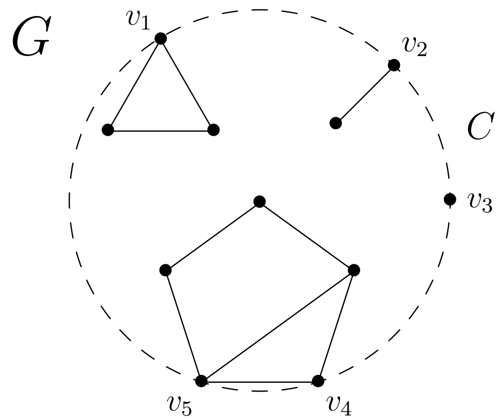


Fig 5. A planar graph meeting C in S_5

Partitions of S_n such as

$$X = (\{v_1, v_3\}, \{v_2, v_4\}, \{v_5\})$$

will be abbreviated in the following (hopefully self-explanatory) manner for clarity and ease:

$$X = (13, 24, 5).$$

We can now define two new chromatic polynomials depending on such partitions:

Definition. The **constrained chromial** of G with respect to X and λ, denoted $K(G, X; \lambda)$, is the number of ways to properly colour G with λ colours subject to the following two conditions:

1. Vertices in the same part of X must receive the same colour.

2. Vertices in different parts of X must receive different colours.

Definition. The **free chromial** of G with respect to X and λ, denoted $F(G, X; \lambda)$, is the number of ways to properly colour G with λ colours subject to only the first condition above.

To formally define the Birkhoff-Lewis we also need the concept of a planar partition (also often called a *noncrossing partition*):

Definition. A partition X of S_n is a **planar partition** if when the vertices of S_n are labeled with increasing integers in cyclic order there do not exist four vertices with $a < b < c < d$ where a and c are in one part but b and d are in another.

Geometrically, X is planar if the convex hulls of each part of X do not intersect.

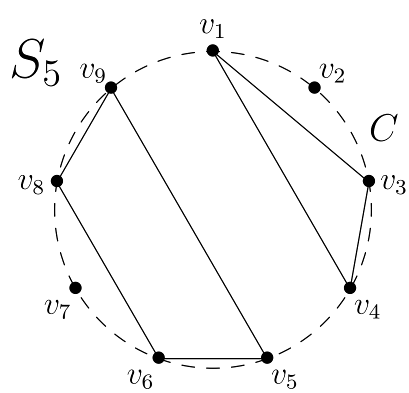


Fig 6. $X = (134, 2, 5689, 7)$ is planar.

Definition. We will call an embedding of a planar graph G a C**-embedding** if G is embedded such that it is bounded by C, with n vertices touching C to form S_n.

Now we are ready for the stars of this paper.

Main Theorem. *For any partition X of S_n and any planar partition Y of S_n, there is a rational function $E(X,Y)$ in λ so that, for any planar graph G and any C-embedding of G,*

$$K(G,X;\lambda) = \sum_{Y planar} E(X,Y)F(G,Y;\lambda). \tag{8}$$

Furthermore, the coefficients $E(X,Y)$ depend only on the partitions X and Y.

Definition. A **Birkhoff-Lewis equation** is any equation of the form (8) and any new relations between free and constrained chromials derived from such equations.

A moment needs to be set aside here to highlight how the absurdly mysterious and beautiful fact tacked on as a "furthermore" in this theorem deserves underscoring. Equations of the form (8) relate free and constrained chromatic polynomials in general, independent of any graphs. Cautis and Jackson [6] dubbed these coefficients **Tutte's chromatic invariants**. If we can find the coefficients for *any* graph meeting C at n vertices, we can use those coefficients for *every* graph meeting C at n vertices. Our proof will in fact take this approach: We will derive a complete set of coefficients

$$\{E(X,Y) : X \text{is any partition of } S_n, Y \text{is a planar partition of } S_n\} \tag{9}$$

from graphs which are essentially empty, then prove they work for all planar graphs using an analogue of the chromatic recurrence (Proposition 2).

4 Proving the Main Theorem

Unfortunately, *essentially* empty graphs are rather more complicated than *the* empty graph. Since we are working with free and constrained chromials the chromatic recurrence does not extend immediately. A little thought should convince the reader that if we contract any edge with one endpoint not on S_n, then the analogue of the chromatic recurrence holds. However, if we contract an edge between two vertices of S_n, we no longer have an S_n in our smaller graph and the statement no longer makes sense (since our free and constrained chromials are defined based on the existence of S_n).

We salvage the chromatic recurrence with the following:

Definition. A **contractive edge** is a relation between between vertices u and v that demands that any colouring assign both u and v the same colour.

Then we redefine G/e to mean replace edge e with a contractive edge. Note how this definition ensures that contractive edges and traditional contraction have the same impact on λ-colourings, and hence the chromatic recurrence still holds with G/e redefined in this way.

Now that we have ensured that S_n is intact, the proof of the chromatic recurrence can be extended to free and constrained chromials to obtain the following:

Remark 3 (Generalized chromatic recurrence)**.**

$$F(G,X;\lambda) = F(G\backslash e,X;\lambda) - F(G/e,X;\lambda) \tag{10}$$

$$K(G,X;\lambda) = K(G\backslash e,X;\lambda) - K(G/e,X;\lambda). \tag{11}$$

Proof. A λ-colouring of $G\backslash e$ with respect to a partition X, whether free or constrained, still splits into two cases: a λ-colouring of G with respect X, or a λ-colouring of G/e with respect to X. Therefore, the proof of the chromatic recurrence extends. □

Definition. Call a C-embedded graph G **basic** if all of its edges are contractive, and all of its components have at least one vertex on C.

With a little work we will find a set of coefficients of the form (9) for all basic graphs. Then we will be in a position to show that same set works for all planar graphs. First we obtain explicit formulas for free and constrained chromials of basic graphs.

The key definition required to compute these chromials is the following:

Definition. The **induced partition** of a basic graph G is the partition Z_G

of S_n where each part is the set of vertices of S_n contained in one component of G.

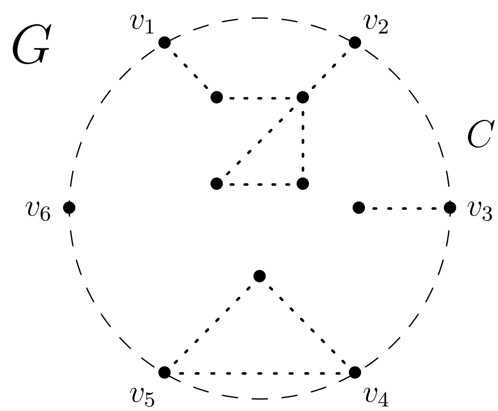


Fig 7. A basic graph with $Z_G = (12, 3, 45, 6)$
(Dotted lines indicate contractive edges)

Note that since G is planar, Z_G is always planar. Furthermore, we can always view a planar partition as the induced partition of some basic graph.

The computation of the free chromial of G relies of the following two concepts, whose significance becomes clear in the next lemma.

Definition. A partition X **refines** a partition Y if every part of X is contained in some part of Y.

Definition. The **chromatic join** of two partitions X and Y is the partition that both X and Y refine which has the most parts. We denote it by $X \vee Y$. For example, if $X = (1, 25, 3, 4, 6)$ and $Y = (1, 23, 46, 5)$, then $X \vee Y = (1, 235, 46)$. Also, let $h(X)$ be the number of parts in X.

Definition. Let $R(X,Y)=1$ if Y refines X and zero otherwise.

Lemma 4. *If G is a basic graph and X is a partition of S_n, then*

$$F(G,X;\lambda)=\lambda^{h(X\vee Z_G)} \tag{12}$$

$$K(G,X;\lambda)=R(X,Z_G)\lambda_{[h(X)]}. \tag{13}$$

Proof of (12). In a free colouring of G with respect to X each cell of X must be monochromatic by definition of a free colouring, and each cell of Z_G must be monochromatic by virtue of being contractive. More succinctly: Any two vertices in the same cell of either X or Z_G must receive the same colour. Hence there are $h(X\vee Z_G)$ sets of vertices which must be coloured, with λ choices of colours for each. The number of ways to do this is

$$P(\overline{K}_{h(X\vee Z_G)};\lambda)=\lambda^{h(X\vee Z_G)}.$$

□

Proof of (13). If Z_G does not refine X, then there are two vertices u and v in different cells of X which are in the same component of G. Being in different cells of X means that u and v receive different colours in any constrained colouring of G. Being in the same component of G means that u and v receive the same colour in any constrained colouring of G (since the edges of G are contractive). Hence there are no such colourings and (13) holds.

If Z_G does refine X, then whenever the parts of X are monochromatic, so are the components of G, hence all we need to worry about is assigning each part of X a unique colour. The number of ways to do this is

$$P(K_{h(X)}; \lambda) = \lambda_{[h(X)]}.$$

$\square$

We now have all the pieces needed to prove the main theorem.

Proof of the Main Theorem. If we enumerate the partitions of S_n as $\{X_1, X_2, \ldots, X_t\}$, then following (8) we can write out the Birkhoff-Lewis equations for a basic graph G explicitly using (12) and (13):

$$R(X_i, Z_G)\lambda_{[h(X_i)]} = \sum_{Y\ planar} E(X_i, Y)\lambda^{h(Y \vee Z_G)} \qquad i = 1, \ldots, t.$$

We want a set of coefficients $E = \{E(X, Y) : X, Y$ planar partitions of $S_n\}$ that work for each set of Birkhoff-Lewis equations of this form—i.e., no matter which basic G that Z_G arises from, the above equations must hold. Since Z_G is necessarily planar, we can remove G from the equations by letting $\{Z_1, \ldots, Z_s\}$ be the set of planar partitions and noting that what we want is a set of coefficients E such that

$$R(X_i, Z_j)\lambda_{[h(X_i)]} = \sum_{Y\ planar} E(X_i, Y)\lambda^{h(Y \vee Z_j)} \tag{14}$$

holds for all pairs (X_i, Z_j)

This system of equations can be recast as a matrix equation by defining matrices D, R, E and M as follows:

$$D_{ij} = \begin{cases} \lambda_{[h(X_i)]} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

$$R_{ij} = R(X_i, Z_j) \tag{16}$$

$$E_{ij} = E(X_i, Z_j) \tag{17}$$

$$M_{ij} = \lambda^{h(Z_i \vee Z_j)} \tag{18}$$

Then $DR = EM$ captures the complete system of equations of the form (14).

Now, Tutte proves that M is always nonsingular over the polynomials [10]. (In the corollary following we address individual values of λ.) Hence $E = DRM^{-1}$ provides promised rational functions for the coefficients for the Birkhoff-Lewis equations for every basic G.

We now prove that the coefficients $E = DRM^{-1}$ also form the Birkhoff-Lewis equations for all other planar graphs. Before diving into an induction on the edges of G that are not contractive, there is one more detail to be taken care of concerning graphs with only contractive edges: what if there are contractive components that do not meet C?

Claim. *E provides the coefficients for any graph whose edgeset is entirely contractive.*

Proof of claim. Let G be a graph whose edges are all contractive. Let H

be the subgraph of G which is basic (i.e., The subgraph consisting of all components of G which touch C). Let k be the number of components of $G\backslash H$. Let X be an arbitrary partition of S_n. For each free or constrained colouring of H with respect to X there are λ^k ways to colour $G\backslash H$ (since colouring k contractive components with no partition restrictions can be done in $P(\overline{K}_k, \lambda) = \lambda^k$ ways). Hence we have

$$F(G, X; \lambda) = \lambda^k F(H, X; \lambda) \tag{19}$$

and

$$K(G, X; \lambda) = \lambda^k K(H, X; \lambda). \tag{20}$$

Now, using (19) and (20) we can transform the Birkhoff-Lewis equations for H into the Birkhoff-Lewis equations for G without affecting the coefficients. We already have

$$K(H, X; \lambda) = \sum_{Y \ planar} E(X, Y) F(H, Y; \lambda)$$

for the basic graph H, which we multiply by λ^k to obtain

$$\lambda^k K(H, X; \lambda) = \sum_{Y \ planar} E(X, Y) \lambda^k F(H, Y; \lambda)$$

and hence

$$K(G, X; \lambda) = \sum_{Y \ planar} E(X, Y)F(G, Y; \lambda),$$

which proves the claim. $\square$

We proceed by induction using the preceding claim as our base case. Let G be a graph with $m > 0$ regular edges, and let e be such an edge. By induction we have

$$K(G\backslash e, X; \lambda) = \sum_{Y} E(X, Y)F(G\backslash e, X; \lambda) \tag{21}$$

$$K(G/e, X; \lambda) = \sum_{Y} E(X, Y)F(G/e, X; \lambda). \tag{22}$$

Adding these together yields

$$K(G\backslash e, X; \lambda) + K(G/e, X; \lambda) = \sum_{Y} E(X, Y)\left(F(G\backslash e, X; \lambda) + F(G/e, X; \lambda)\right)$$

Then, using (10) and (11), this becomes

$$K(G, X; \lambda) = \sum_{Y} E(X, Y)F(G, Y; \lambda)$$

and hence E provides the coefficients for every C-embedding of every planar graph. $\square$

As a bonus, our proof provides a recipe for computing Birkhoff-Lewis equations: Compute $E = DRM^{-1}$ to obtain the coefficients for the Birkhoff-Lewis equations. Furthermore, this computation would work even if we do not take all t partitions as our X_i's—any subset will do. Together with the following strengthening of the theorem, we obtain the concrete Birkhoff-Lewis equations which we seek.

Corollary 5. *If λ is not a zero of the first n Chebyshev polynomials, then $E(X,Y)$ can be evaluated at λ in the main theorem.*

Proof. In [5], Cautis and Jackson give the determinant of M for S_n as a product of the first n Chebyshev Polynomials to a particular power. $\square$

5 Shrinking the System

The previous section provides a recipe for obtaining the Birkhoff-Lewis equations. There are a few problems though. For starters, the systems quickly become huge. In the cases of $S_4, S_5,$ and S_6 the number of planar partitions are 14, 42, 132 respectively. Since these numbers define dimensions of M, we see that even just in the case of the 6-ring the computation is rather horrid in practice (though none of these are impossible: even without this succinct approach, the 4-ring and the 5-ring were worked out by Birkhoff and Lewis in their original paper [4], and the 6-ring was later tackled by Hall and Lewis [7]).

Since we're developing the Birkhoff-Lewis equations in the context of the four-colour theorem, we may assume that any graph we embed inside C is a triangulation, so no matter what vertices touch C to form S_n they must form C_n—a cycle of length n—which follows the cyclic order of S_n. We call this special cycle $\mathcal{C}_n$.

Having $\mathcal{C}_n$ means that we have a lot more information about the free and constrained chromials of the graphs we are interested in—namely that any chromial for a partition which contains an edge in one of its parts is zero.

Above, our basic graphs are graphs whose edges are all contractive and whose components all meet C in at least one vertex. We can obtain a much smaller system by considering a more complex base case: call a graph **semi-basic** if it is the union of a basic graph and $\mathcal{C}_n$.

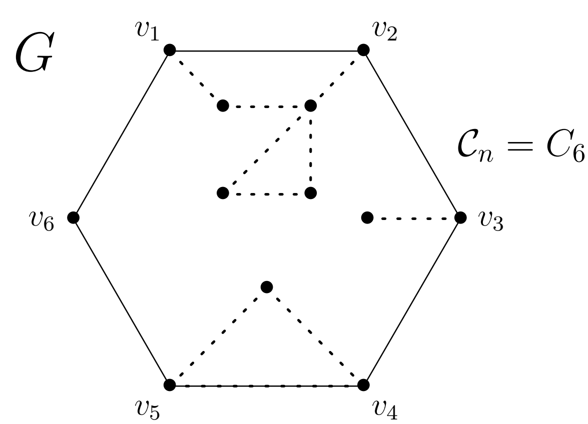


Fig 8. A semi-basic graph with $Z_G = (12, 3, 45, 6)$

If we could find coefficients E for all semi-basic graphs, then we could use the same inductive steps as the proof of the main theorem to conclude that E was the set of coefficients for any planar graph bounded by $\mathcal{C}_n$. Naturally this is a weaker existence proof than the one above, but the recipe that arises from this form of the proof will prove to be computationally more manageable.

Remark 6. Computing E for semi-basic graphs provides the coefficients for any planar graph bounded by $\mathcal{C}_n$

Proof. A semi-basic graph can be obtained from any planar graph G bounded by $\mathcal{C}_n$ by contracting and deleting every edge of G except those of $\mathcal{C}_n$. Hence, the induction in the proof of the Main Theorem can be modified to use semi-basic graphs as its base case. $\square$

Our approach earlier relied on having explicit expressions for evaluating free and constrained chromials of basic graphs. We will achieve something similar for semi-basic graphs.

Definition. A partition X will be called **simple** if no part of X contains an edge of the bounding $\mathcal{C}_n$.

Let $\overline{R}(X, Y) = 1$ if Y refines X and both X and Y are simple partitions, and 0 otherwise.

Lemma 7. *If G is a semi-basic graph and X is a partition of S_n, then*

$$F(G, X; \lambda) = \overline{R}(X, Z_G) F(X \vee Z_G, C_n; \lambda) \tag{23}$$

$$K(G, X; \lambda) = \overline{R}(X, Z_G) \lambda_{[h(X)]}. \tag{24}$$

Proof of (23). If X is not simple, then $F(G, X; \lambda) = 0$ on the left since an edge in some part of X has endpoints which cannot receive the same colour. $\overline{R}(X, Z_G) = 0$ ensures we also get zero on the right.

If X is simple, we are again in a position where any two vertices in the same cell of either X or Z_G must receive the same colour. Hence a free colouring of G respecting X must be a free colouring of G respecting $X \vee Z_G$.

Then, since Z_G captures the only impact $G \backslash \mathcal{C}_n$ has on colouring—i.e., the requirement that its contractive components be monochromatic, we have that a colouring is a free colouring of G with respect to $X \vee Z_G$ if and only it is a free colouring of C_n with respect to $X \vee Z_G$. $\square$

Proof of (24). If X is not simple, then as above $K(G, X; \lambda) = 0$ on the left, and $\overline{R}(X, Z_G) = 0$ ensures we also get zero on the right.

If X is simple, then a colouring is a constrained colouring of G if and only if it is a colouring of $G \backslash C_n$ (which is a basic graph) since adjacent vertices of C are in different parts of X. Therefore, by (13) we have

$$K(G, X; \lambda) = R(X, Z_G)\lambda_{[h(X)]}.$$

Since $R(X, Y) = \overline{R}(X, Y)$ for simple partitions, this completes the proof. $\square$

With these expressions for the free and constrain chromials in hand we can state our goal in a way similar to (14): we want a set of coefficients E such that

$$\overline{R}(X_i, Z_j)\lambda_{[h(X_i)]} = \sum_{Y \ planar} E(X_i, Y)F(Y \vee Z_j, C_n; \lambda) \tag{25}$$

holds for all pairs (X_i, Z_j) of partitions with Z_j planar.

From here we can see that any time either X_i or Z_j is not simple, this equation is the trivial statement $0 = 0$ and hence can be ignored. Then (25) need only consider pairs (X_i, Z_j) of simple partitions. This yields a much smaller system to contend with (S_4, S_5 and S_6 give rise to M's with dimensions 3, 6 and 15 equations respectively—as opposed to 14, 42 and 132).

If we take *any* set $\{X_1, \ldots, X_t\}$ of simple partitions of $\mathcal{C}_n$, and we let $\{Z_1, \ldots, Z_s\}$ be the set of all simple planar partitions of $\mathcal{C}_n$ this system of equations can be recast as a matrix equation by defining matrices D, R, E and M as follows:

$$D_{ij} = \begin{cases} \lambda_{[h(X_i)]} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

$$R_{ij} = \overline{R}(X_i, Z_j) = R(X_i, Z_j) \tag{27}$$

$$E_{ij} = E(X_i, Z_j) \tag{28}$$

$$M_{ij} = F(C_n, X_i \vee Z_j; \lambda) \tag{29}$$

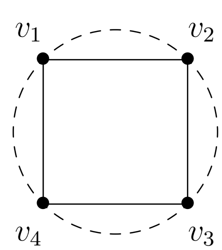
Then $DR = EM$ captures the complete system of equations of the form (25). Hence, to complete this system all we need to do is compute the entries of M.

Again, we can solve this system if we can invert M. The question remains open whether there is some way to guarantee in general the non-singularity of M in the simplified Birkhoff-Lewis system. All we can guarantee at present is that for S_4, S_5 and S_6, M is invertible over the polynomials, and with $\lambda = 4$ since we computed these matrices explicitly.

Remark 8. We close this section by pointing out that any subset of the edges of $\mathcal{C}_n$ can in the same way be used to define a class of semi-basic graphs: the free chromials become easier to compute with fewer edges on the circle, and the constrained chromials can be computed for such graphs in the same way as in Lemma 7 (with a modified $\overline{R}$). The benefits or penalties associated with the different M's arising from different concepts of semi-basic deserve to be explored.

6 Computing Birkhoff-Lewis Equations

We walk through a derivation of the simplified Birkhoff-Lewis system for a graph embedded in $\mathcal{C}_4$. We need the simple partitions of $\mathcal{C}_4$:



$X_1 = (1, 2, 3, 4)$
$X_2 = (13, 2, 4)$
$X_3 = (24, 1, 3)$
$X_4 = (13, 24)$

Fig 9. Simple partitions of $\mathcal{C}_4$

Note that these coincide with the partitions induced by 4-colourings given in Figure 1 at the beginning of the paper. Using these partitions, we construct the matrices D and R as in (26) and (28) using only planar partitions:

$$D = \begin{bmatrix} \lambda(\lambda-1)(\lambda-2)(\lambda-3) & 0 & 0 \\ 0 & \lambda(\lambda-1)(\lambda-2) & 0 \\ 0 & 0 & \lambda(\lambda-1)(\lambda-2) \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Then, to compute M, we note that we have the following relationships amongst the joins of these partitions:

$$\begin{aligned} X_i \vee X_i &= X_i && \forall i \\ X_i \vee X_j &= X_j \vee X_i && \forall i \\ X_1 \vee X_i &= X_i && \forall i \\ X_i \vee X_4 &= X_4 && \forall i \\ X_2 \vee X_3 &= X_4 \end{aligned}$$

This implies that if we compute $F(\mathcal{C}_4, X_i; \lambda)$ for $i = 1, 2, 3, 4$ we will have all the entries of M.

In practice these free chromatic polynomials are obtained by computing the regular chromatic polynomial of the graph obtained by contracting the cells of X_i.

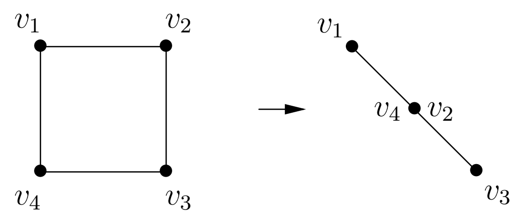


Fig 10. Contracting the cells of X_2 to obtain a path with chromatic polynomial $\lambda(\lambda-1)^2$

Both X_2 and X_3 give rise to the graph in Figure 10. X_1 gives C_4, which we saw in the Chromatic Polynomials section (Section 2) has chromial $\lambda(\lambda-1)(\lambda^2-3\lambda+3)$, and X_4 leaves K_2, which has chromial $\lambda(\lambda-1)$. So we have

$$M_4 = \begin{bmatrix} \lambda(\lambda-1)(\lambda^2-3\lambda+3) & \lambda(\lambda-1)^2 & \lambda(\lambda-1)^2 \\ \lambda(\lambda-1)^2 & \lambda(\lambda-1)^2 & \lambda(\lambda-1) \\ \lambda(\lambda-1)^2 & \lambda(\lambda-1) & \lambda(\lambda-1)^2 \end{bmatrix}$$

By whatever means necessary, we obtain M^{-1}. Then, with $D, R,$ and M in hand we multiply to obtain:

$$E = \begin{bmatrix} \frac{\lambda(\lambda-3)}{\lambda^2-3\lambda+1} & -\frac{(\lambda-1)(\lambda-3)}{\lambda^2-3\lambda+1} & -\frac{(\lambda-1)(\lambda-3)}{\lambda^2-3\lambda+1} \\ \frac{1}{\lambda^2-3\lambda+1} & \frac{(\lambda-1)(\lambda-3)}{\lambda^2-3\lambda+1} & -\frac{\lambda-2}{\lambda^2-3\lambda+1} \\ \frac{1}{\lambda^2-3\lambda+1} & -\frac{\lambda-2}{\lambda^2-3\lambda+1} & \frac{(\lambda-1)(\lambda-3)}{\lambda^2-3\lambda+1} \end{bmatrix}.$$

The Main Theorem tells us that we can write out Birkhoff-Lewis equations for any $\mathcal{C}_n$-embedded graph using the entries of E as coefficients. To express the resulting equations more cleanly, we multiply through by $\lambda^2-3\lambda+1$ and use F_i and K_i for $F(G, X_i; \lambda)$ and $K(G, X_i; \lambda)$ respectively.

Birkhoff-Lewis equations for any graph embedded in $\mathcal{C}_4$.

$$(\lambda^2 - 3\lambda + 1)K_1 = \lambda(\lambda - 3)F_1 + (\lambda - 1)(\lambda - 3)F_2 + (\lambda - 1)(\lambda - 3)F_3 \quad (30)$$

$$(\lambda^2 - 3\lambda + 1)K_2 = F_1 + (\lambda - 1)(\lambda - 3)F_2 - (\lambda - 2)F_3 \quad (31)$$

$$(\lambda^2 - 3\lambda + 1)K_3 = F_1 - (\lambda - 2)F_2 + (\lambda - 1)(\lambda - 3)F_3 \quad (32)$$

Remark 9. Any free chromial can be expressed trivially as a sum of constrained chromials in the following way:

$$F_i = \sum_{j \text{ s.t. } X_i \text{ refines } X_j} K_j.$$

Specifically we have the following relationships for any $\mathcal{C}_4$-embedded graph:

$$F_1 = K_1 + K_2 + K_3 + K_4 \quad (33)$$

$$F_2 = K_2 + K_4 \quad (34)$$

$$F_3 = K_3 + K_4 \quad (35)$$

Then, substituting (33) to (35) into (30) to (32) and solving this system with $\lambda = 4$ we obtain

$$K_1 - K_2 - K_3 + 2K_4 = 0,$$

which is the Birkhoff-Lewis equation used Proposition 1.

We now investigate Birkhoff-Lewis equations for $\mathcal{C}_5$.

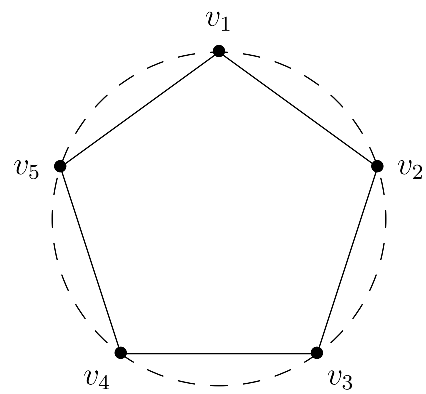
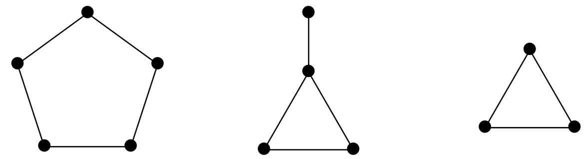


Fig 11. $\mathcal{C}_5$

These are the simple partitions of $\mathcal{C}_5$, divided into types (i.e., same partition up to rotation):

Planar	*Nonplanar*
$X_1 = (1, 2, 3, 4, 5)$	$\ldots$
$\ldots$	$X_7 = (24, 35, 1)$
$X_2 = (14, 2, 3, 5)$	$X_8 = (14, 35, 2)$
$X_3 = (25, 1, 3, 4)$	$X_9 = (14, 25, 3)$
$X_4 = (13, 2, 4, 5)$	$X_{10} = (13, 52, 4)$
$X_5 = (24, 1, 3, 5)$	$X_{11} = (13, 24, 5)$
$X_6 = (35, 1, 2, 4)$	

We will again index our system with only the planar partitions. There are only three distinct graphs arising from contracting according to chromatic joins of planar partitions of $\mathcal{C}_5$:



G_1 G_2 G_3

Fig 12. Graphs used to compute $F(\mathcal{C}_5, X_i \vee X_j; \lambda)$

We have

$$\alpha = P(G_1;\lambda) = \lambda(\lambda-1)^4 - \lambda(\lambda-1)^3 + \lambda(\lambda-1)(\lambda-2)$$
$$\beta = P(G_2;\lambda) = \lambda(\lambda-1)^2(\lambda-2)$$
$$\gamma = P(G_3;\lambda) = \lambda(\lambda-1)(\lambda-2),$$

from which follows our matrix of chromatic joins:

$$M_5 = \begin{bmatrix} \alpha & \beta & \beta & \beta & \beta & \beta \\ \beta & \beta & \gamma & 0 & 0 & \gamma \\ \beta & \gamma & \beta & \gamma & 0 & 0 \\ \beta & 0 & \gamma & \beta & \gamma & 0 \\ \beta & 0 & 0 & \gamma & \beta & \gamma \\ \beta & \gamma & 0 & 0 & \gamma & \beta \end{bmatrix}$$

Computing DRM^{-1} give us the following matrix of coefficients:

$$(\lambda^2 - 3\lambda + 1)E =$$

$$\begin{bmatrix} (\lambda+1)(\lambda-4) & -(\lambda+1)(\lambda-4) & -(\lambda+1)(\lambda-4) & -(\lambda+1)(\lambda-4) & -(\lambda+1)(\lambda-4) & -(\lambda+1)(\lambda-4) \\ 2 & \lambda^2-5\lambda+5 & -\lambda-2 & -1 & -1 & -\lambda-2 \\ 2 & -\lambda-2 & \lambda^2-5\lambda+5 & -\lambda-2 & -1 & -1 \\ 2 & -1 & -\lambda-2 & \lambda^2-5\lambda+5 & -\lambda-2 & -1 \\ 2 & -1 & -1 & -\lambda-2 & \lambda^2-5\lambda+5 & -\lambda-2 \\ 2 & -\lambda-2 & -1 & -1 & -\lambda-2 & \lambda^2-5\lambda+5 \end{bmatrix}$$

For $\lambda = 4$, after making the substitutions suggested by Remark 9, we simplify and obtain the following Birkhoff-Lewis equations:

$$K_1 = 0 \tag{36}$$

$$-K_2 + K_5 - K_7 + K_8 = 0 \tag{37}$$

$$-K_4 + K_6 - K_7 + K_{11} = 0 \tag{38}$$

$$-K_3 - K_4 + K_5 + K_6 - K_7 + K_{10} = 0 \tag{39}$$

$$-K_2 - K_3 + K_5 + K_6 - K_7 + K_9 = 0 \tag{40}$$

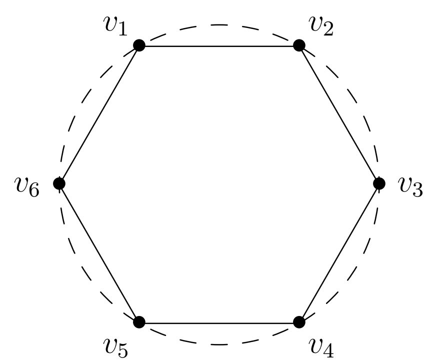


Fig 13. $\mathcal{C}_6$

For the computation of E for $\mathcal{C}_6$, we use more than just the planar partitions (this has the dual benefit of demonstrating a non-square E and providing the equation we need in the next section). We will find coefficients to express the constrained chromials with respect to any simple partition (as opposed to any simple planar partition in the last example) as a combination of free chromials. Here is the enumeration of simple partitions we use:

First the planar simple partitions (divide once again according to rotational symmetry):

$$\begin{array}{lll}
X_1 = (1,2,3,4,5,6) & X_5 = (15,2,3,4,6) & X_{11} = (135,2,4,6) \\
\ldots & X_6 = (26,1,3,4,5) & X_{12} = (246,1,3,5) \\
X_2 = (36,1,2,4,5) & X_7 = (13,2,4,5,6) & \ldots \\
X_3 = (14,2,3,5,6) & X_8 = (24,1,3,5,6) & X_{13} = (15,24,3,6) \\
X_4 = (25,1,3,4,6) & X_9 = (35,1,2,4,6) & X_{14} = (26,35,1,4) \\
\ldots & X_{10} = (46,1,2,3,5) & X_{15} = (13,46,2,5)
\end{array}$$

Then the nonplanar simple partitions:

$$\begin{array}{lll}
X_{16} = (15, 36, 2, 4) & X_{25} = (15, 46, 2, 3) & X_{34} = (13, 46, 25) \\
X_{17} = (26, 14, 3, 5) & X_{26} = (15, 26, 3, 4) & \quad\ldots \\
X_{18} = (13, 25, 4, 6) & X_{27} = (26, 13, 4, 5) & X_{35} = (135, 24, 6) \\
X_{19} = (24, 36, 1, 5) & X_{28} = (24, 13, 5, 6) & X_{36} = (246, 35, 1) \\
X_{20} = (35, 14, 2, 6) & X_{29} = (35, 24, 6, 1) & X_{37} = (135, 46, 2) \\
X_{21} = (46, 25, 1, 3) & X_{30} = (46, 35, 1, 2) & X_{38} = (246, 15, 3) \\
\quad\ldots & \quad\ldots & X_{39} = (135, 26, 4) \\
X_{22} = (25, 36, 1, 4) & X_{31} = (14, 25, 36) & X_{40} = (246, 13, 5) \\
X_{23} = (14, 36, 2, 5) & \quad\ldots & \quad\ldots \\
X_{24} = (14, 25, 3, 6) & X_{32} = (15, 24, 36) & X_{41} = (135, 246) \\
\quad\ldots & X_{33} = (26, 35, 14) &
\end{array}$$

We leave the overly enthusiastic reader to confirm that there are 11 distinct graphs arising from contracting cells of chromatic joins of planar partitions of $\mathcal{C}_6$. Computing their free chromials yields:

$$a = \lambda(\lambda-1)^5 - \lambda(\lambda-1)^4 + \lambda(\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2)$$

$$b = \lambda(\lambda-1)^2(\lambda-2)^2$$

$$c = \lambda(\lambda-1)^4 - \lambda(\lambda-1)^2(\lambda-2)$$

$$d = \lambda(\lambda-1)^3$$

$$e = \lambda(\lambda-1)^3$$

$$f = \lambda(\lambda-1)^2(\lambda-2) - \lambda(\lambda-1)(\lambda-2)$$

$$g = \lambda(\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2)$$

$$h = \lambda(\lambda-1)^2(\lambda-2) - \lambda(\lambda-1)(\lambda-2)$$

$$i = \lambda(\lambda-1)^2$$

$$j = \lambda(\lambda-1)$$

$$k = \lambda(\lambda-1)(\lambda-2)$$

which then gives

$$M_6 = \begin{bmatrix} a & b & b & b & c & c & c & c & c & c & d & d & e & e & e \\ b & b & f & f & h & 0 & 0 & h & 0 & 0 & 0 & 0 & k & 0 & 0 \\ b & f & b & f & 0 & h & 0 & 0 & h & 0 & 0 & 0 & 0 & k & 0 \\ b & f & f & b & 0 & 0 & h & 0 & 0 & h & 0 & 0 & 0 & 0 & k \\ c & h & 0 & 0 & c & g & d & e & d & g & d & i & e & i & i \\ c & 0 & h & 0 & g & c & g & d & e & d & i & d & i & e & i \\ c & 0 & 0 & h & d & g & c & g & d & e & d & i & i & i & e \\ c & h & 0 & 0 & e & d & g & c & g & d & i & d & e & i & i \\ c & 0 & h & 0 & d & e & d & g & c & g & d & i & i & e & i \\ c & 0 & 0 & h & g & d & e & d & g & c & i & d & i & i & e \\ d & 0 & 0 & 0 & d & i & d & i & d & i & d & j & i & i & i \\ d & 0 & 0 & 0 & i & d & i & d & i & d & j & d & i & i & i \\ e & k & 0 & 0 & e & i & i & e & i & i & i & i & e & j & j \\ e & 0 & k & 0 & i & e & i & i & e & i & i & i & j & e & j \\ e & 0 & 0 & k & i & i & e & i & i & e & i & i & j & j & e \end{bmatrix}$$

From here E can be computed, but since neither the massive matrix of coefficients nor the huge set of equations obtained are particularly illuminating to stare at, we omit them here.

7 Reducing Configurations

The following configuration was named in honour of Birkhoff:

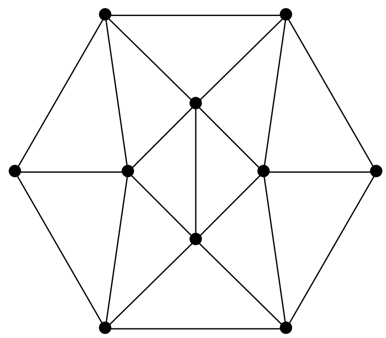


Fig 14. The Birkhoff Diamond

We use this configurations to demonstrate the potential to do reductions with the Birkhoff-Lewis equations.

Proposition 4. *The Birkhoff Diamond is a reducible configuration.*

Proof. Let G be a minimal counter-example embedded in the plane. Suppose to the contrary that G contains the Birkhoff Diamond as a subgraph. Let G' be the subgraph of G obtained by deleting the four vertices inside the diamond.

Viewing the outer ring of the Diamond as $\mathcal{C}_6$, we see that Birkhoff-Lewis equations for $\mathcal{C}_6$ must hold for G'. In the system arising from M_6 of the previous section, we can solve for K_{41} to obtain:

$$K_{41} = \frac{K_{13}}{2} - \frac{K_{25}}{2} - \frac{K_{29}}{2} + \frac{K_{36}}{2} + \frac{K_{37}}{2}. \tag{41}$$

Now the colourings that induce $X_{13}, X_{25}, X_{29}, X_{36},$ and X_{37} all extend in to the Birkhoff Diamond. The following figure demonstrates this by taking a colouring of the ring induced by X_i, and showing a colouring of the Birkhoff Diamond that uses it.

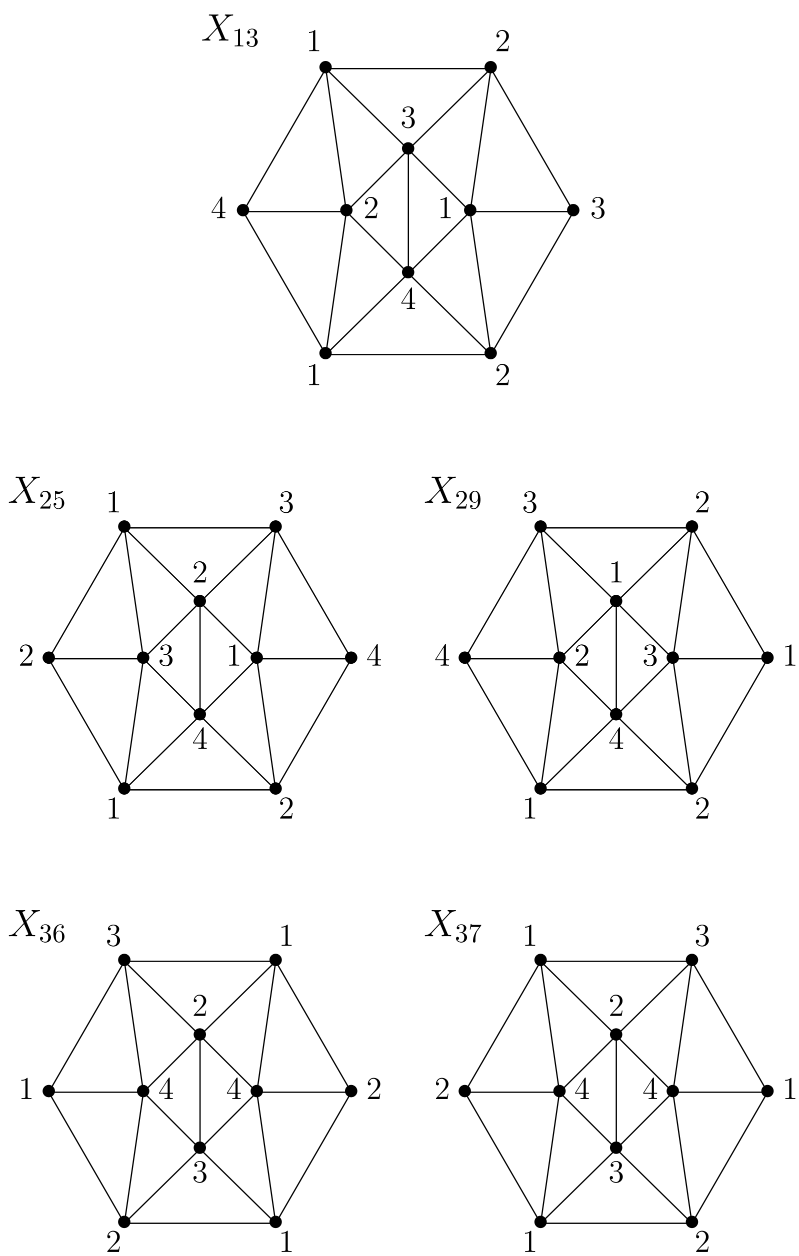


Fig 15. Colouring partitions that *extend in* to the Birkhoff Diamond

This implies that $K_{13}, K_{25}, K_{29}, K_{36},$ and K_{37} are all zero (else we could extend that 4-colouring of G' to a colouring of G). Then, by (41), we also get $K_{41} = 0$.

Now consider replacing the Birkhoff Diamond with the following configuration (recall that dotted lines are contractive edges) to obtain a graph $\overline{G}$:

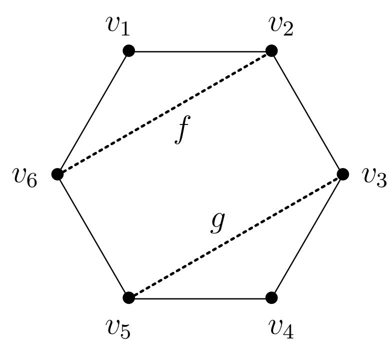


Fig 16. $\overline{G}$

Note first that if $\overline{G}$ had a loop, then we would have the following 4-separation in G (indicated by the boxed vertices):

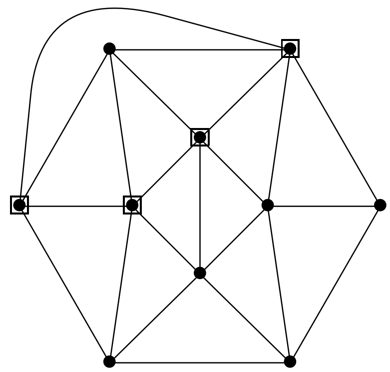


Fig 17. A loop in $\overline{G}$ implies a 4-separation in G

This is because a loop would have to arise from contracting the endpoints of an edge in G—that is, there needs to be an edge that shares endpoints with either f or g. Thanks to the symmetry of f and g and the fact G is a triangulation, either situation is covered by Figure 17.

Therefore $\overline{G}$ is a loopless planar graph which is smaller than G in which any 4-colouring must assign the same colours to the pairs $\{v_2, v_6\}$ and $\{v_3, v_5\}$ of $\mathcal{C}_6$. This implies that to 4-colour $\overline{G}$ we must have a colouring of G' which induces a partition that is refined by $(26, 35, 1, 4)$. These partitions are: $X_{14}, X_{33}, X_{36}, X_{39}$ and X_{41}. We have shown that $K_{41} = 0$ above, and it can be verified as in Figure 15 that $K_{14}, K_{33}, K_{36}, K_{39}$ all induce colourings that extend in to the Birkhoff Diamond and are hence zero.

Therefore the fact that G is not 4-colourable implies our new, smaller $\overline{G}$ is not 4-colourable, and hence the Birkhoff Diamond is a reducible configuration. $\square$

8 Internally 6-connected Counter-examples

In this section we shall prove using Birkhoff-Lewis equations that any 5-separation in a minimal counter-example consists of the neighbours of a vertex of degree five—that is, any minimal counter-example is internally 6-connected.

We will work with the simple partitions of $\mathcal{C}_5$ under different labels, labels that anchor the constrained polynomials to the vertices of $\mathcal{C}_5$ in a very nice way. We let $Y_{(i,j)}$ be the simple partition with a single cell of size greater than one which contains the vertices v_i and v_j. Similarly, we let $Y_{(k)}$ be the partition with exactly one cell of size one which contains the vertex v_k. Here is the explicit relabeling:

Planar	*Nonplanar*
$Y_0 = X_1 = (1,2,3,4,5)$	$\dots$
$\dots$	$Y_{(1)} = X_7 = (24,35,1)$
$Y_{(1,4)} = X_2 = (14,2,3,5)$	$Y_{(2)} = X_8 = (14,35,2)$
$Y_{(2,5)} = X_3 = (25,1,3,4)$	$Y_{(3)} = X_9 = (14,25,3)$
$Y_{(1,3)} = X_4 = (13,2,4,5)$	$Y_{(4)} = X_{10} = (13,52,4)$
$Y_{(2,4)} = X_5 = (24,1,3,5)$	$Y_{(5)} = X_{11} = (13,24,5)$
$Y_{(3,5)} = X_6 = (35,1,2,4)$	

We then naturally use $K_{(i,j)}$ and $K_{(k)}$ for the constrained chromials associated with these newly labeled partitions. Without too much effort, we can obtain the following equations from 36 to 40:

$$K_0 = 0 \tag{42}$$

$$K_{(1)} + K_{(1,4)} = K_{(2)} + K_{(2,4)} \tag{43}$$

$$K_{(2)} + K_{(2,5)} = K_{(3)} + K_{(3,5)} \tag{44}$$

$$K_{(3)} + K_{(3,1)} = K_{(4)} + K_{(4,1)} \tag{45}$$

$$K_{(4)} + K_{(4,2)} = K_{(5)} + K_{(5,2)} \tag{46}$$

$$K_{(5)} + K_{(5,3)} = K_{(1)} + K_{(1,3)} \tag{47}$$

Note that there are still only four linearly independent equations here.

We will use the following notation: for $i \in \{1, \ldots, 5\}$ and $k \in \{0, \ldots, 4\}$ we write

$$i \oplus k = \begin{cases} i + k & \text{if } i + k \leq 5 \\ i + k - 5 & \text{otherwise} \end{cases}$$

We can then rewrite (43) to (47) as

$$K_{(i)} + K_{(i,i\oplus 3)} = K_{(i\oplus 1)} + K_{(i\oplus 1,i\oplus 3)} \quad i = 1, \ldots, 5 \tag{48}$$

Proposition 5. *A minimal counter-example is internally 6-connected.*

Proof. Let G be a minimal counter-example and suppose to the contrary G

has a 5-separation which leaves two components larger than a single vertex when deleted. Again, this 5-separation forms a cycle since G is a triangulation. Call the cycle $\mathcal{C}_5$. Suppose we have an embedding of G and let H^{in} be the subgraph obtained by deleting the vertices embedded outside $\mathcal{C}_5$ and let H^{out} be defined analogously.

We denote that a property holds for both K^{in} and K^{out} terms by replacing them by K terms.

Claim. *The following properties hold for the constrained chromials of H^{in} and H^{out}:*

$$K_{(1)} + K_{(2)} + K_{(3)} + K_{(4)} + K_{(5)} \geq 1 \tag{49}$$

$$K^{in}_{(i)} K^{out}_{(i)} = 0 \text{ and } K^{in}_{(i,i\oplus 1)} K^{out}_{(i,i\oplus 1)} = 0 \quad i = 1, \ldots, 5 \tag{50}$$

$$K_{(i)} + K_{(i\oplus 2, i\oplus 4)} + K_{(i\oplus 4, i\oplus 1)} + K_{(i\oplus 1, i\oplus 3)} \geq 1 \tag{51}$$

Proof of (49). Since H^{in} and H^{out} can be interchanged without loss of generality, it suffices to prove the claim for H^{out}. Suppose $K^{out}_{(1)} + K^{out}_{(2)} + K^{out}_{(3)} + K^{out}_{(4)} + K^{out}_{(5)} = 0$. Then there is no way to colour H^{out} that uses fewer than 4 colours on the ring (since these are all the constrained chromials associated with partitions that have fewer than four cells). This implies we could replace H^{in} with a vertex v of degree 5 whose neighbours are the vertices of $\mathcal{C}_5$

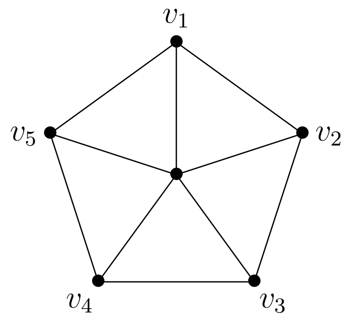


Fig 18. A potential replacement for H^{in}

to obtain a graph smaller than G which cannot be four coloured (since we do not have any colour left to colour v) for a contradiction. $\square$

Proof of (50). If equation (50) fails, there would exist a 4-colouring of H^{in} and a 4-colouring of H^{out} which we could glue together to obtain a 4-colouring of G. Hence, equation (50) must hold. $\square$

Proof of (51). Again, it suffices to prove the claim for H^{out}. Furthermore, due to the rotational symmetry, we may assume *wlog* $i = 1$. Suppose to the contrary we have

$$K^{out}_{(1)} + K^{out}_{(3,5)} + K^{out}_{(5,2)} + K^{out}_{(2,4)} = 0. \tag{52}$$

Let G' be the graph obtained by replacing H^{in} with the edges $\{v_1, v_3\}$ and $\{v_1, v_4\}$.

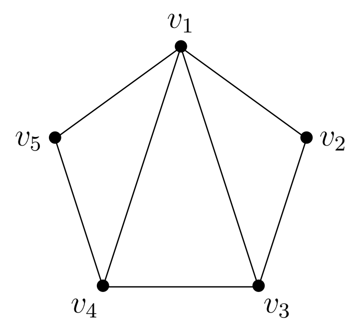


Fig 19. Another potential replacement for H^{in}

Note that now any colouring that induces partition

$$X_{(2)}, X_{(3)}, X_{(4)}, X_{(5)}, X_{(1,3)} \text{ or } X_{(1,4)}$$

cannot give a 4-colouring of G' due to these two edges. Since (52) states that there is no way to colour the rest of G' and induce any of the remaining partitions, we conclude that G' has no 4-colourings. Since it is smaller than G, this is a contraction. $\square$

Claim. *Suppose*

$$K^{in}_{(3)} = K^{in}_{(4)} = 0 \textit{ and } K^{in}_{(2)} > 0.$$

Then

$$K^{in}_{(1,4)} = K^{in}_{(1,3)} = 0 \textit{ and } K^{in}_{(1)} = 0.$$

Proof. From (45), we have $K^{in}_{(4,1)} = K^{in}_{(3,1)}$ since we suppose $K^{in}_{(3)} = K^{in}_{(4)} = 0$.

We also have $K^{out}_{(2)} = 0$ from (50) since $K^{in}_{(2)} > 0$.

From (44), we have $K^{in}_{(3,5)} > 0$ since $K^{in}_{(3)} = 0$ and $K^{in}_{(2)} > 0$. This in turn implies that $K^{out}_{(3,5)} = 0$, again by (50).

From (51) we have

$$K^{out}_{(2)} + K^{out}_{(4,1)} + K^{out}_{(1,3)} + K^{out}_{(3,5)} \geq 1,$$

but since we have $K^{out}_{(2)} = 0$ and $K^{out}_{(3,5)} = 0$, we know that either $K^{out}_{(4,1)}$ or $K^{out}_{(1,3)}$ is positive.

Then, by (50) either $K^{in}_{(4,1)}$ or $K^{in}_{(1,3)}$ is zero. But since $K^{in}_{(4,1)} = K^{in}_{(1,3)}$, we have $K^{in}_{(4,1)} = K^{in}_{(1,3)} = 0$.

Finally, by (47), we have $K^{in}_{(1)} > 0$ since $K^{in}_{(1,3)} = 0$ and $K^{in}_{(5,3)} > 0$. $\square$

Now, by (50), either

$$|\{i \mid K^{in}_{(i)} = 0,\ i = 1, \ldots, 5\}| \geq 3 \tag{53}$$

$$\text{or} \quad |\{i \mid K^{out}_{(i)} = 0,\ i = 1, \ldots, 5\}| \geq 3 \tag{54}$$

As before we may assume we are the case of (53). We know that there exists $i \in \{1, \ldots, 5\}$ such that $K^{in}_{(i)} > 0$. Due to rotational symmetry, we may also assume that

$$K^{in}_{(4)} = K^{in}_{(3)} = 0 \text{ and } K^{in}_{(2)} \geq 0.$$

Claim 8 then gives us

$$K^{in}_{(1,3)} = K^{in}_{(1,4)} = 0,\ K^{in}_{(1)} > 0.$$

Also, since we have (53), we get $K^{in}_{(5)} = 0$.

Relabel $1, 2, 3, 4, 5$ by $2, 1, 5, 4, 3$ respectively and apply Claim 8 again. Then we see that (in terms of our original labelling)

$$K^{in}_{(2,4)} = K^{in}_{(2,5)} = 0.$$

Finally, we have by (51) that

$$K^{in}_{(5)} + K^{in}_{(3,1)} + K^{in}_{(1,4)} + K^{in}_{(4,2)} \geq 1.$$

Since all the terms on the left here are zero, we have a contradiction. $\square$

9 Future Directions

The main lingering question here is can the differences between the various forms of the Birkhoff-Lewis equations (obtained by choosing a different subset of the partitions to obtain equations for, or modifying the basic graph used to obtain the equations) be pinned down? Knowing what benefits or detriments are present in a given set of Birkhoff-Lewis equations would have obvious advantages.

The other path that deserves some exploration involves the nonsingularity of various semi-basic graphs and the desire to automate the computation of the M matrices built from them.

References

[1] K. Appel and W. Haken, *Every planar map is four colorable. I. Discharging*, Illinois J. Math. **21** (1977), 429–490.

[2] K. Appel, W. Haken, and J. Koch, *Every planar map is four colorable. II. Reducibility*, Illinois J. Math. **21** (1977), 491–567.

[3] G. D. Birkhoff, *The Reducibility of Maps*, Amer. J. Math. **35** (1913), 115–128.

[4] G. D. Birkhoff and D. C. Lewis, *Chromatic polynomials*, Trans. Amer. Math. Soc. **60** (1946), 355–451.

[5] S. Cautis and D. M. Jackson, *The matrix of chromatic joins and the temperley-lieb algebra.*, J. Comb. Theory, Ser. B **89** (2003), no. 1, 109–155.

[6] S. Cautis and D. M. Jackson, *On tutte's chromatic invariant*, Draft (2005).

[7] D. W. Hall and D. C. Lewis, *Coloring six-rings*, Trans. Amer. Math. Soc. **64** (1948), 184–191.

[8] A. B. Kempe, *On the Geographical Problem of the Four Colours*, Amer. J. Math. **2** (1879), 193–200.

[9] N. Robertson, D. Sanders, P. Seymour, and R. Thomas, *The four-colour theorem*, J. Combin. Theory Ser. B **70** (1997), 2–44.

[10] W. T. Tutte, *On the Birkhoff-Lewis equations*, Discrete Math. **92** (1991), 417–425.

[11] ______, *The Birkhoff-Lewis equations for graph-colorings*, Quo vadis, graph theory?, Ann. Discrete Math., vol. 55, North-Holland, Amsterdam, 1993, pp. 153–158.

[12] Douglas B. West, *Introduction to graph theory*, Prentice Hall Inc., New Jersey, 1996.

[13] Robert A. Wilson, *Graphs, colourings and the four-colour theorem*, Oxford University Press, Oxford, 2002.

A Discharging

This summary of discharging owes a lot to the very pleasant treatment in [13].

So-called "Discharging" is the method used to generate unavoidable sets. Speaking very broadly, we can sum the method up as follows:

Given a graph G

1. Place a "charge" at each vertex of G.
2. Sum up the charge on all vertices of G to obtain the total charge.
3. Describe a *discharging algorithm*—a set of instructions for moving charge between vertices of G in such a way that conserves total charge.

4. Argue that certain structural properties of G would force the total charge to change under the discharging algorithm (in spite of conservation), and hence G cannot have those structural properties.

We can apply this method directly to proving that a set of configurations U is unavoidable by making the "structural properties" of step 4 the hypothesis that G contains no configuration in U. We demonstrate with an example. Define a set of configurations as follows:

$$\begin{aligned} U = \{&\text{a vertex of degree 5 with a neighbour of degree 5,} \\ &\text{a vertex of degree 5 with a neighbour of degree 6}\}. \end{aligned}$$

Proposition 6. *U is unavoidable.*

Proof. Let G be a minimal counter-example, and suppose it is embedded in the plane (we need this to talk about its faces). As before, we may assume G is an internally 6-connected triangulation.

Place a charge of value $6 - d(u)$ at each vertex $u \in V(G)$. Note that all vertices of degree 5 have a charge of 1, vertices of degree 6 have no charge, and vertices of higher degree have negative charge.

We can calculate the total charge on the graph using Euler's Formula: Let $v, e,$ and f be the number of vertices, edges and faces of G respectively. Note that since G is a triangulation, $2e = 3f$, and hence $6f - 4e = 0$ (an observation whose utility will be clear in the sum below). We sum up the charges

$$\begin{aligned}\sum_{u\in V}(6-d(v)) &= 6v-\sum_{u\in V}d(v)\\ &= 6v-2e\\ &= 6v-2e+(6f-4e)\\ &= 6(v-e+f)\\ &= 12 \qquad \text{(by Euler's Formula)}\end{aligned}$$

[Side note: even before doing any discharging we have learned something about minimum counter-examples: since this sum is nonnegative, any minimum counter-example must have vertices of degree five. Using this same general method, the discharging algorithm which does nothing proves that

$$\bar{U} = \{\text{a vertex of degree five}\}$$

is an unavoidable set. This is in fact the unavoidable set Kempe used in his famously flawed proof of the 4CT [8].]

We will use the following discharging algorithm:

Algorithm. *Each vertex of degree 5 gives a charge of $\frac{1}{5}$ to each of its neighbours which has degree at least 7.*

Note that this algorithm conserves the total charge on the graph. Now we suppose to the contrary that no subgraph of G is a configuration in U (i.e., that no adjacent vertices of G both have degrees five, or have degrees five and six).

This implies that every neighbour of every vertex of degree 5 has degree seven or greater, so every vertex of degree five now has charge zero (each one discharges $\frac{1}{5}$ to each of its neighbours). Furthermore, the vertices of degree 6 are unchanged by this discharging algorithm and hence still have no charge.

Now consider a vertex w of degree $k \geq 7$. Since G is a triangulation, the k neighbours of w form a cycle. No two consecutive neighbours on this cycle have degree five by assumption. This implies that at most $\frac{1}{2}k$ neighbours of w have degree five, and hence w gains at most $\frac{1}{10}k$ charge from the algorithm. The new charge for w is thus at most

$$6 + k - \frac{1}{10}k = 6 - \frac{9}{10}k \leq 6 - \frac{9}{10}7 < 0.$$

Now we have that every vertex of G has nonpositive charge, and so the total charge, in spite of conservation, is no longer 12. This is the contradiction we seek. $\square$