Comprehensive Exam – Discrete Optimization – Spring 2001

1. [20 marks]

   (i) State the Integral Max-Flow Min-Cut Theorem.

   (ii) Let $G = (V, E)$ be a bipartite graph with bipartition (P, Q) and let $d \in \mathbf{Z}_+^V$. Use the Integral Max–Flow Min–Cut Theorem to derive necessary and sufficient conditions on G for the existence of a spanning subgraph H of G such that the degree of v in H is d_v, for all $v \in V$.

2. [25 marks]
   (In this question you may use standard results on matching.) Let $G = (V, E)$ be an undirected graph, and let $\mathcal{I}$ denote the family of all subsets I of V such that there is a matching of G that saturates each node in I.

   (i) Prove that $\mathcal{I}$ is the family of independent sets of a matroid M on V.

   (ii) Prove that the rank function r of M satisfies, for each $A \subseteq V$

   $$r(A) = \min_{X \subseteq V} |A| + |X| - \mathrm{odd}_A(G - X),$$

   where $\mathrm{odd}_A(G - X)$ denotes the number of odd components of $G - X$ whose node-sets are contained in A.

3. [15 marks]
   Let P denote the polyhedron $\{x \in \mathbf{R}^n : Ax \leq b\}$ where $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$. A *face* of P is a set of the form $F = \{x \in P : c^T x = c_0\}$ for some valid inequality $c^T x \leq c_0$ for P. Prove that F is a face of P if and only if there is a subsystem $A'x \leq b'$ of $Ax \leq b$ such that $F = \{x \in P : A'x = b'\}$. (You may use the linear programming duality theorem.)

4. [20 marks]
   Let M be a matroid on the set S with rank-function r, and let P denote the polytope $\{x \in \mathbf{R}_+^S : x(A) \leq r(A) \text{ for all } A \subseteq S\}$.

   (i) Prove that P is the convex hull of incidence vectors of independent sets of M.

   (ii) For $A \subseteq S$, let F_A denote the face $\{x \in P : x(A) = r(A)\}$ of P. Prove that, if F_A is a facet of P then $r(A) < r(C) + r(A - C)$ for all proper non-empty subsets C of A.

5. [20 marks]

Consider the problem (P): "Given an undirected graph $G = (V, E)$ and positive integer k, does G have a stable set of size at least k?". (A stable set is a set of mutually nonadjacent nodes of G.)

(i) Show that (P) is $\mathcal{NP}$-complete by reducing the satisfiability problem to it.

(ii) Explain briefly why, when (P) is modified to replace "graph" by "bipartite graph", the resulting problem is in $\mathcal{P}$.